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On Uniformly Convex Functions

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In this paper we study uniformly convex functions and uniformly convex functions at a point, giving some properties and characterizations of them. Further, we give some examples and applications of these types of functions.

INTRODUCTION

In 1966 Polyak [1] introduced the notions of strongly convex and strongly quasi-convex functions. In 1976 Rockafellar [2] studied the strongly convex functions in connection with the proximal point algorithm. In 1978 there appeared two interesting papers of Vladimirov *et al.* [3, 4], where they introduced and studied uniformly convex and uniformly quasi-convex functions, respectively, which differ from strongly convex and strongly quasi-convex functions in the sense of Polyak. Important examples of uniformly convex and uniformly quasi-convex functions are furnished in these papers. Vial [5] studied some geometrical properties of strongly convex and strongly quasi-convex functions in the sense of Polyak in finite dimensions.

In this paper we study the uniformly convex functions giving some characterizations and examples of such functions. We also study the uniformly convex functions at a point, functions which are in connection with the wellset problems in the sense of Tyhonov (see Zolezzi [6]) and the strong convergence of the proximal point algorithm [2]. We also mention that in the literature the notion of uniform convexity appeared with an equivalent meaning (see Ilioi [7]) or a different one (see Ciorănescu [8]).

The paper is divided into four sections and an Appendix. In the first section we give the basic notions, notations and preliminary results which we need. The second section is devoted to the study of uniformly convex functions and uniformly convex functions at a point. In the third section we present the one-dimensional case. Some examples of uniformly convex functions are given in Section 4. The Appendix is devoted to some results we need in the paper, results that are not necessarily all known or results—as in Proposition A.3—that together with others in the first part help one to

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understand the relations between the properties of the space X and functions of type $x \to \psi(\|x\|)$, where ψ: $[0, \infty[\to [0, \infty]$ is a convex function. We shall frequently use the results in the Appendix, so we suggest that it is better to read the Appendix after Section 1.

1. Notations, Definitions and Preliminary Results

Throughout this paper X denotes a real Banach space and X^* its dual. If $x^* \in X^*$ and $x \in X$, $x^*(x)$ is denoted by $\langle x, x^* \rangle$ or $\langle x^*, x \rangle$. If $A \subset X$, $\operatorname{int} A$, $\operatorname{cl} A$, $\operatorname{co} A$, $\overline{\operatorname{co}}\, A$ denotes the interior, the closure, the convex hull and the closed convex hull of A, respectively. R denotes the reals and R_+, R_+^*, $\bar{R}_+$ denote the nonnegative reals, positive reals and $R_+ \cup \{\infty\}$, respectively. If M, N are subsets of X, $a \in X$, $\lambda \in R$ then $M + N = \{x + y: x \in M, y \in N\}$, $M \sim N = \{x \in M: x \notin N\}$, $\lambda M = \{\lambda x: x \in M\}$, $M + a = M + \{a\}$; in a similar way $M - N$ and $M - a$ are defined. The strong and weak convergence are denoted by $\to$ and $\rightharpoonup$, respectively.

The following notations, definitions and results are well known in convex analysis and can be found in [9] for the finite dimensional case and in [10] for the infinite dimensional case. For a function $f: X \to R \cup \{\infty\}$, the *domain* is the set $\operatorname{dom} f = \{x \in X: f(x) < \infty\}$, the *epigraph* is the set $\operatorname{epi} f = \{(x, a) \in X \times R: f(x) \leqslant a\}$. f is *proper* if $\operatorname{dom} f \neq \varnothing$ ($\varnothing$ = the empty set). f is *convex* if

$$f(\lambda x + (1-\lambda) y) \leqslant \lambda f(x) + (1-\lambda) f(y), \qquad \forall x, y \in \operatorname{dom} f, \quad x \neq y, \quad \forall \lambda \in]0, 1[. \tag{1.1}$$

f is *strictly convex* if the inequality is strict in (1.1). It is known that for a convex function $f: X \to R \cup \{\infty\}$ there is the *directional derivative* at every $x \in \operatorname{dom} f$, defined by

$$f'(x; y) = \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t} \in R \cup \{-\infty, +\infty\}. \tag{1.2}$$

The function $f'(x; \cdot): X \to R \cup \{-\infty, +\infty\}$ is sublinear if we make the conventions $-\infty + \infty = \infty$ and $0.\ \infty = 0$. If $X = R$ and $x \in \operatorname{dom} f$ there are the following notations: $f'_+(x) = f'(x, 1)$ and $f'_-(x) = f'(x, -1)$. Following Rockafellar [9] in the case $X = R$, we make the conventions $f'_+(x) = f'_-(x) = -\infty$ at the left of $\operatorname{dom} f$ and $f'_+(x) = f'_-(x) = \infty$ at the right of $\operatorname{dom} f$. For the function $f: X \to R \cup \{\infty\}$, we denote by $\overline{\operatorname{co}}\, f$ the function whose epigraph is $\overline{\operatorname{co}}(\operatorname{epi} f)$. The *subdifferential* of f at $\bar{x} \in \operatorname{dom} f$ is the set

$$\partial f(\bar{x}) = \{x^* \in X^*: \langle x - \bar{x}, x^* \rangle \leqslant f(x) - f(\bar{x})\ \forall x \in X\}. \tag{1.3}$$

If $\bar{x} \notin \operatorname{dom} f$ then $\partial f(\bar{x}) = \varnothing$. It follows immediately that $\partial f(\bar{x}) = \partial f'(\bar{x}; 0)$. If the convex function f is continuous at $\bar{x} \in \operatorname{dom} f$ [or f is lower semicontinuous (l.s.c.) and $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$] then $\partial f(\bar{x})$ is a nonempty w^*-compact convex subset of X^* and

$$f'(\bar{x}, y) = \max\{\langle y, x^* \rangle : x^* \in \partial f(\bar{x})\}, \qquad \forall y \in X. \tag{1.4}$$

Therefore, if $\bar{x} \in \operatorname{dom} f$ is a continuity point for f, then $f'(\bar{x}, \cdot)$ is a sublinear continuous functional. The *conjugate* of $f: X \to R \cup \{\infty\}$ is the l.s.c. convex function $f^*: X^* \to R \cup \{-\infty, +\infty\}$,

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in X\}. \tag{1.5}$$

$f^{**}: X \to R \cup \{-\infty, +\infty\}$ is defined similarly. $f^{**} = f$ iff f is a l.s.c. convex function. The *indicator function* of $U \subset X$ is the function $I_U: X \to R \cup \{\infty\}$, $I_U(x) = 0$ for $x \in U$ and $I_U(x) = \infty$ for $x \in X \sim U$; I_U is l.s.c. and convex iff $U = \overline{\operatorname{co}}\, U$. By ∂f we mean the set $\{(x, x^*) \in X \times X^* : x^* \in \partial f(x)\}$; $\operatorname{dom} \partial f = \{x \in X : \partial f(x) \neq \varnothing\}$ $(\subset \operatorname{dom} f)$, $R(\partial f) = \{x^* \in X^* : \exists x \in X$ with $x^* \in \partial f(x)\}$ $(\subset \operatorname{dom} f^*)$, and $\partial f^* = (\partial f)^{-1}$ if f is a proper l.s.c. convex function. For the following definitions see [11] or [8]. Let X be Banach space and let $S_X = \{x \in X : \|x\| = 1\}$. $S(x, \varepsilon) = \{y \in X : \|y - x\| < \varepsilon\}$, $\bar{S}(x, \varepsilon) = \{y \in X : \|y - x\| \leqslant \varepsilon\}$. X is *strictly convex* if $\forall x, y \in S_X$, $x \neq y : \|x + y\| < 2$; X is *locally uniformly convex* if

$$\forall x \in S_X\ \forall \varepsilon \in]0, 2]\ \exists \delta \in]0, 2[\ \forall y \in S_X, \|y - x\| \geqslant \varepsilon : \|x + y\| \leqslant 2 - \delta, \tag{1.6}$$

and X is *uniformly convex* if

$$\forall \varepsilon \in]0, 2]\ \exists \delta \in]0, 2[\ \forall x, y \in S_X, \|y - x\| \geqslant \varepsilon : \|x + y\| \leqslant 2 - \delta. \tag{1.7}$$

X is *smooth* if $\forall x \in S_X\ \exists x^* \in S_{X^*}$ unique such that $\langle x, x^* \rangle = 1$.

2. Uniformly Convex Functions

Throughout this section X is a Banach space and $f: X \to R \cup \{\infty\}$ is a proper l.s.c. convex function whose domain is not a singleton.

Definition 2.1. f is *uniformly convex* (u.c.) *at* $x \in \operatorname{dom} f$ if there exists $\delta: R_+ \to \bar{R}_+$ with $\delta(t) > 0$ for $t > 0$ such that

$$f(\lambda x + (1 - \lambda) y) \leqslant \lambda f(x) + (1 - \lambda) f(y) - \lambda(1 - \lambda)\, \delta(\|y - x\|), \qquad \forall y \in \operatorname{dom} f, \quad \forall \lambda \in]0, 1[. \tag{2.1}$$

Remark 2.1. In Definition 2.1 we can replace (2.1) by any of the following relations:

$$\begin{aligned}&\forall\varepsilon>0\ \exists\delta>0\ \forall y\in\operatorname{dom}f,\ \|y-x\|\geqslant(>,=)\varepsilon,\ \forall\lambda\in\,]0,1[:\\&f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y)-\delta\lambda(1-\lambda),\end{aligned}\tag{2.1$'$}$$

or

$$\begin{aligned}&\forall\varepsilon>0\ \exists\delta>0\ \forall y\in\operatorname{dom}f,\ \|y-x\|\geqslant(>,=)\varepsilon:\\&f\left(\frac{x+y}{2}\right)\leqslant\frac{1}{2}f(x)+\frac{1}{2}f(y)-\delta.\end{aligned}\tag{2.1$''$}$$

Taking into account Proposition A.1(vi) all the variants of (2.1$'$) and (2.1$''$) are equivalent, respectively. Thus, (2.1) $\Leftrightarrow$ (2.1$'$) $\Rightarrow$ (2.1$''$). Suppose (2.1$''$) holds and take $y\in\operatorname{dom}f$, $\|y-x\|\geqslant\varepsilon$, and $\lambda\in\,]0,\frac{1}{2}]$. Then

$$\begin{aligned}f(\lambda x+(1-\lambda)y)&=f\left(2\lambda\left(\frac{x+y}{2}\right)+(1-2\lambda)y\right)\\&\leqslant 2\lambda f\left(\frac{x+y}{2}\right)+(1-2\lambda)f(y)\\&\leqslant\lambda f(x)+\lambda f(y)-2\lambda\delta+(1-2\lambda)f(y)\\&\leqslant\lambda f(x)+(1-\lambda)f(y)-2\lambda(1-\lambda)\delta.\end{aligned}$$

Similarly for $\lambda\in\,]\frac{1}{2},1[$, so that (2.1$''$) $\Rightarrow$ (2.1$'$).

Remark 2.2. The best δ in Definition 2.1 is nondecreasing with $\operatorname{int}(\operatorname{dom}\delta)\neq\varnothing$.

Consider $F=\{\psi: R_+\to\bar{R}_+:\psi$ is l.s.c., convex, $\operatorname{int}(\operatorname{dom}\psi)\neq\varnothing$, $\psi(t)=0\Rightarrow t=0\}$.

THEOREM 2.1. *Let X be reflexive and $\bar{x}\in\operatorname{dom}f$ and $\bar{x}^*\in\partial f(\bar{x})\neq\varnothing$. Then among the following conditions the implications* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) *are valid.*

(i) *f is u.c. at $\bar{x}$,*

(ii) $\exists\psi\in F\ \forall x\in X; f(x)\geqslant f(\bar{x})+f'(\bar{x};x-\bar{x})+\psi(\|x-\bar{x}\|)$,

(iii) $\exists\psi\in F\quad\forall x\in X,\quad\forall x^*\in\partial f(\bar{x}): f(x)\geqslant f(\bar{x})+\langle x-\bar{x},x^*\rangle+\psi(\|x-\bar{x}\|)$,

(iv) $\exists\psi\in F\ \forall x\in X: f(x)\geqslant f(\bar{x})+\langle x-\bar{x},\bar{x}^*\rangle+\psi(\|x-\bar{x}\|)$,

(v) $\exists\psi\in F\quad\forall x^*\in X^*:\quad f^*(x^*)\leqslant f^*(\bar{x}^*)+\langle\bar{x},x^*-\bar{x}^*\rangle+\psi^*(\|x^*-\bar{x}^*\|)$,

(vi) *f^* is Fréchet differentiable at $\bar{x}^*$,*

(vii) $\forall \psi \in F\ \forall (x, x^*) \in \partial f$: $\langle x - \bar{x}, x^* - \bar{x}^* \rangle \geqslant \psi(\|x - \bar{x}\|)$,

(viii) $\exists \varphi$: $R_+ \to \bar{R}_+$ *nondecreasing, with* $\varphi_+(0) = \lim_{t \downarrow 0} \varphi(t) = 0$ *such that*

$$\forall (x, x^*) \in \partial f\colon \|x - \bar{x}\| \leqslant \varphi(\|x^* - \bar{x}^*\|).$$

Proof. (i) $\Rightarrow$ (ii). Take $x \in \operatorname{dom} f$; from (2.1) we have

$$\frac{f(\bar{x} + \lambda(x - \bar{x})) - f(\bar{x})}{\lambda} \leqslant f(x) - f(\bar{x}) - (1 - \lambda)\, \delta(\|x - \bar{x}\|).$$

Taking the limit as $\lambda \downarrow 0$ we obtain

$$f(x) \geqslant f(\bar{x}) + f'(\bar{x}; x - \bar{x}) + \delta(\|x - \bar{x}\|). \tag{2.2}$$

Take $\mu(t) = \inf\{f(x) - f(\bar{x}) - f'(\bar{x}; x - \bar{x}) \colon x \in \operatorname{dom} f,\ \|x - \bar{x}\| = t\}$. It is clear from (2.2) that $\mu(t) \geqslant \delta(t)$. Let us show that μ satisfies

$$\mu(ct) \geqslant c\mu(t), \qquad \forall c \geqslant 1,\ t \geqslant 0. \tag{2.3}$$

Let $c > 1$, $t > 0$ and $x \in \operatorname{dom} f$, $\|x - \bar{x}\| = ct$, and take $y = (1 - (1/c))\bar{x} + (1/c)x \in \operatorname{dom} f$; then $\|y - \bar{x}\| = t$. From the convexity of f we have $f(y) - f(\bar{x}) \leqslant (f(x) - f(\bar{x}))/c$, so that $f(y) - f(\bar{x}) - f'(\bar{x}, y - \bar{x}) \leqslant (f(x) - f(\bar{x}) - f'(\bar{x}, x - \bar{x}))/c$. Therefore $c \cdot \mu(t) \subseteq \mu(ct)$, i.e., (2.3) holds. Taking $\psi = \overline{\mathrm{co}}\, \mu$, from Proposition A.5, $\psi \in F$, so that (ii) is valid.

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(v) $\Rightarrow$ (iv). Let $x \in X$. Since f, ψ are proper l.s.c. convex functions, we have $f = f^{**}$, $\psi = \psi^{**}$ so that

$$\begin{aligned} f(x) = f^{**}(x) &= \sup\{\langle x, x^* \rangle - f^*(x^*) \colon x^* \in X^*\} \\ &\geqslant \sup\{\langle x, x^* \rangle - f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle \\ &\quad - \psi^*(\|x^* - \bar{x}^*\|) \colon x^* \in X^*\} \\ &= \sup\{\langle x - \bar{x}, x^* - \bar{x}^* \rangle - \psi^*(\|x^* - \bar{x}^*\|) \colon \\ &\quad x^* \in X^*\} + \langle x, \bar{x}^* \rangle - f^*(\bar{x}^*) \\ &= \psi(\|x - \bar{x}\|) + \langle x, \bar{x}^* \rangle - f^*(\bar{x}^*). \end{aligned}$$

But $f^*(\bar{x}^*) + f(\bar{x}) = \langle \bar{x}, \bar{x}^* \rangle$, so that $f(x) \geqslant f(\bar{x}) + \langle x - \bar{x}, \bar{x}^* \rangle + \psi(\|x - \bar{x}\|)$. Therefore (iv) holds.

(iv) $\Rightarrow$ (vii). Let $(x, x^*) \in \partial f$; then

$$f(\bar{x}) \geqslant f(x) + \langle \bar{x} - x, x^* \rangle,$$
$$f(x) \geqslant f(\bar{x}) + \langle x - \bar{x}, \bar{x}^* \rangle + \psi(\|x - \bar{x}\|),$$

so that $\langle x - \bar{x}, x^* - \bar{x}^* \rangle \geqslant \psi(\|x - \bar{x}\|)$.

(vii) $\Rightarrow$ (viii). From (vii) we have

$$\|x - \bar{x}\| \cdot \|x^* - \bar{x}^*\| \geqslant \langle x - \bar{x}, x^* - \bar{x}^* \rangle \geqslant \psi(\|x - \bar{x}\|),$$

so that

$$\|x^* - \bar{x}^*\| \geqslant \frac{\psi(\|x - \bar{x}\|)}{\|x - \bar{x}\|} = \tilde{\varphi}(\|x - \bar{x}\|), \tag{2.4}$$

where $\tilde{\varphi}: R_+ \to \bar{R}_+$, $\tilde{\varphi}(t) = \psi(t)/t$ for $t > 0$, $\tilde{\varphi}(0) = 0$. It is known [see also Proposition A.1(i)] that $\tilde{\varphi}$ is nondecreasing. Let $\varphi: R_+ \to R_+$ be defined by $\varphi(t) = \max\{\tau \geqslant 0: \tilde{\varphi}(\tau) \leqslant t\}$. Since $\tilde{\varphi}(t) = 0 \Rightarrow t = 0$, by Proposition A.2, φ is a nondecreasing function with $\varphi_+(0) = 0$. From (2.4) and the definition of φ we have $\|x - \bar{x}\| \leqslant \varphi(\|x^* - \bar{x}^*\|)$, i.e., (viii) holds.

(viii) $\Rightarrow$ (v). Let $T = \sup \operatorname{dom} \varphi > 0$ and $x^* \in \operatorname{dom} \partial f^* = R(\partial f)$ such that $\|x^* - \bar{x}^*\| < T$; there exist ε, $\delta > 0$ such that for all $y^* \in S(x^*, \varepsilon)$ we have $\|y^* - x^*\| < T - \delta$. Therefore the set $\{y \in X: (y^*, y) \in \partial f^*, y^* \in S(x^*, \varepsilon)\}$ is bounded. Applying [12, Theorem 1], it follows $x^* \in \operatorname{int}(\operatorname{dom} \partial f^*) \subset \operatorname{dom} f^*$. Therefore $S(\bar{x}^*, T) \subset \operatorname{dom} f^*$, so that for $x^* \in S(\bar{x}^*, T)$, $f^{*\prime}(x^*, u^*) = \max\{\langle x, u^* \rangle: x \in \partial f^*(x^*)\} = \max\{\langle x, u^* \rangle: (x, x^*) \in \partial f\}$. But, for $(x, x^*) \in \partial f$, we have

$$\langle x, u^* \rangle = \langle x - \bar{x}, u^* \rangle + \langle \bar{x}, u^* \rangle \leqslant \|x - \bar{x}\| \, \|u^*\| + \langle \bar{x}, u^* \rangle$$
$$\leqslant \|u^*\| \, \varphi(\|x^* - \bar{x}^*\|) + \langle \bar{x}, u^* \rangle,$$

so that

$$f^{*\prime}(\bar{x}^* + t(x^* - \bar{x}^*); x^* - \bar{x}^*) \leqslant \|x^* - \bar{x}^*\| \, \varphi(t \, \|x^* - \bar{x}^*\|) + \langle \bar{x}, x^* - \bar{x}^* \rangle.$$

Integrating on $[0, 1]$ with respect to t, we obtain

$$f^*(x^*) - f^*(\bar{x}^*) \leqslant \int_0^1 \|x^* - \bar{x}^*\| \, \varphi(t \, \|x^* - \bar{x}^*\|) \, dt + \langle \bar{x}, x^* - \bar{x}^* \rangle$$

$$= \int_0^{\|x^* - \bar{x}^*\|} \varphi(t) \, dt + \langle \bar{x}, x^* - \bar{x}^* \rangle$$

$$= \tilde{\psi}(\|x^* - \bar{x}^*\|) + \langle \bar{x}, x^* - \bar{x}^* \rangle,$$

where $\tilde{\psi}(t) = \int_0^t \varphi(\tau)\, d\tau$. In the case $T = \infty$, the above relation is valid for every $x^* \in X^*$. If $T < \infty$, let x^* such that $\|x^* - \bar{x}^*\| = T$. It is clear that $\lim_{t \uparrow T} \tilde{\psi}(t) = \tilde{\psi}(T)$; let $x_n^* \to x^*$, $\|x_n^* - \bar{x}^*\| < T$. Then

$$f^*(x_n^*) - f^*(\bar{x}^*) \leqslant \tilde{\psi}(\|x_n^* - \bar{x}^*\|) + \langle \bar{x}, x_n^* - \bar{x}^* \rangle.$$

Since f^* is l.s.c., it follows that

$$f^*(x^*) - f^*(\bar{x}^*) \leqslant \tilde{\psi}(\|x^* - \bar{x}^*\|) + \langle \bar{x}, x^* - \bar{x}^* \rangle.$$

If $\|x^* - \bar{x}^*\| > T$ then $\tilde{\psi}(\|x^* - \bar{x}^*\|) = \infty$, so that we must only show that $\tilde{\psi}^* \in F$, which is obvious by Proposition A.2(ii) since $\tilde{\psi}'_+(0) = \varphi_+(0) = 0$.

(v) $\Rightarrow$ (vi). From (v) we have that $\bar{x}^* \in \operatorname{int}(\operatorname{dom} f^*)$. Since $\bar{x} \in \partial f^*(\bar{x}^*)$ we have

$$0 \leqslant f^*(x^*) - f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle \leqslant \psi^*(\|x^* - \bar{x}^*\|),$$

so that

$$\limsup_{x^* \to \bar{x}^*} \left| \frac{f^*(x^*) - f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle}{\|x^* - \bar{x}^*\|} \right| \leqslant \lim_{t \downarrow 0} \frac{\psi^*(t)}{t} = \psi_+^{*\prime}(0) = 0,$$

by Proposition A.2(ii). Hence (vi) holds.

(vi) $\Rightarrow$ (v). Let $\tilde{\psi}(t) = \sup\{f^*(x^*) - f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle : \|x^* - \bar{x}^*\| = t\}$. Let us show that $\tilde{\psi}$ is convex; let $0 \leqslant t_1 < t_2$ and $\lambda \in\,]0, 1[$ and suppose that $\tilde{\psi}(t_1)$, $\tilde{\psi}(t_2) < \infty$. Take $x^* \in X^*$ such that $\|x^* - \bar{x}^*\| = \lambda t_1 + (1 - \lambda) t_2 = t$. For $x_1^* = \bar{x}^* + (t_1/t)(x^* - \bar{x}^*)$, $x_2^* = \bar{x}^* + (t_2/t)(x^* - \bar{x}^*)$ we have $\|x_1^* - \bar{x}^*\| = t_1$, $\|x_2^* - \bar{x}^*\| = t_2$ and $x^* = \lambda x_1^* + (1 - \lambda) x_2^*$. So

$$\begin{aligned} f^*(x^*) &- f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle \\ &\leqslant \lambda (f^*(x_1^*) - f(\bar{x}^*) - \langle \bar{x}, x_1^* - \bar{x}^* \rangle) \\ &\quad + (1 - \lambda)(f^*(x_2^*) - f^*(\bar{x}^*) - \langle \bar{x}, x_2^* - \bar{x}^* \rangle) \\ &\leqslant \lambda \tilde{\psi}(t_1) + (1 - \lambda)\, \tilde{\psi}(t_2). \end{aligned}$$

Therefore $\tilde{\psi}(\lambda t_1 + (1 - \lambda) t_2) \leqslant \lambda \tilde{\psi}(t_1) + (1 - \lambda)\, \tilde{\psi}(t_2)$. From the definition of $\tilde{\psi}$ we have

$$f^*(x^*) - f^*(\bar{x}^*) - \langle \bar{x}, x^* - \bar{x}^* \rangle \leqslant \tilde{\psi}(\|x^* - \bar{x}^*\|).$$

Since f^* is Fréchet differentiable at $\bar{x}^*$ it follows that $\lim_{t \downarrow 0}(\tilde{\psi}(t)/t) = \tilde{\psi}'_+(0) = 0$. Taking $\psi = \tilde{\psi}^*$, (v) is proved. ∎

COROLLARY 2.1. *Let X be a reflexive Banach space. The following conditions are equivalent for $(\bar{x}, \bar{x}^*) \in \partial f$:*

(α) $\exists \psi \in F$, $M > 0$ *such that*

$$f(x) \geqslant f(\bar{x}) + \langle x - \bar{x}, \bar{x}^* \rangle + \psi(\|x - \bar{x}^*\|), \qquad \forall x \in S(\bar{x}, M),$$

(β) $\exists \psi \in F$, $M > 0$ *such that*

$$f^*(x^*) \leqslant f^*(\bar{x}^*) + \langle \bar{x}, x^* - \bar{x}^* \rangle + \psi^*(\|x^* - \bar{x}^*\|), \qquad \forall x^* \in S(\bar{x}^*, M),$$

(γ) $\exists \varphi: R_+ \to \bar{R}_+$ *nondecreasing*,

with $\varphi_+(0) = 0$ *and* $M > 0$ *such that*

$$\|x - \bar{x}\| \leqslant \varphi(\|x^* - \bar{x}^*\|), \qquad \forall (x, x^*) \in \partial f, \quad x^* \in S(\bar{x}^*, M).$$

The proof reduces to the preceding theorem taking appropriate extensions for $\psi|[0, M/2]$ or $\varphi|[0, M/2]$, respectively.

Remark 2.3. For $\psi(t) = at^2$, $a > 0$ (or $\varphi(t) = at$, $a > 0$) in Corollary 2.1 one obtains [12, Proposition 7].

COROLLARY 2.2. *If f is u.c. at any $x \in \operatorname{dom} f$ then $R(\partial f)$ is open, f^* is Fréchet differentiable on* $\operatorname{int}(\operatorname{dom} f^*) = R(\partial f)$ *and* $\partial f^* = \nabla f^*$ *is continuous on* $\operatorname{int}(\operatorname{dom} f^*)$. *Moreover, f^* is Fréchet equidifferentiable on $\partial f(x)$ for every $x \in \operatorname{dom}(f)$.*

Proof. The assertion of the corollary follows from Theorem 2.1(vi) and (viii). ∎

Remark 2.4. If $\bar{x}$ is a solution of the problem

$$(\mathscr{P}) \qquad \inf_{x \in X} f(x),$$

then the condition (iv) with $\bar{x}^* = 0$ is nothing else but the condition of well posedness of problem ($\mathscr{P}$) as in Zolezzi [6]. Some implications in Theorem 2.1 are similar to some of those appearing in [6].

COROLLARY 2.3. *Suppose $\bar{x}$ is a solution for ($\mathscr{P}$). If for $(\bar{x}, 0)$ some of the conditions* (i)-(viii) *in Theorem* 2.1 *are verified, then every minimizing sequence converges strongly to $\bar{x}$. Moreover, if $f(x_n) \to f(\bar{x})$ then*

$$\|x_n - \bar{x}\| \leqslant \psi^{-1}(f(x_n) - f(\bar{x})),$$

where $\psi \in F$.

Proof. If some of the conditions (i)-(iii) are verified, then (iv) is also verified so that $f(x) \geqslant f(\bar{x}) + \psi(\|x - \bar{x}\|)$, $\forall x \in \operatorname{dom} f$. So $f(x_n) \geqslant f(\bar{x}) + \psi(\|x_n - \bar{x}\|)$, from which the conclusion follows. ∎

Remark 2.5. If one of the conditions (i)–(viii) in Theorem 2.1 is verified then the proximal point algorithm [12] converges strongly. These conditions are weaker than the Lipschitz continuity of $(\partial f)^{-1}$ at 0.

Indeed the proximal point algorithm [12] yields a minimizing sequence, which, by Corollary 2.3, converges strongly.

DEFINITION 2.2. (i) f is *uniformly convex* (u.c.) if there exists δ: $R_+ \to \bar{R}_+$ with $\delta(t) > 0$ for $t > 0$ such that

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\,\delta(\|y - x\|),\qquad \forall x, y \in \operatorname{dom} f, \quad \forall \lambda \in\,]0, 1[, \tag{2.5}$$

(ii) f is *u.c.* on $U \subset X$ if $f + I_U$ is u.c.

(iii) the function μ: $R_+ \to \bar{R}_+$,

$$\mu(t) = \inf\{(\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y))/\lambda(1-\lambda):\ x, y \in \operatorname{dom} f, \|x - y\| = t, \lambda \in\,]0, 1[\} \tag{2.6}$$

is called the *exact modulus of convexity of f.*

Remark 2.6. In Definition 2.2(i) we can replace (2.5) by any of the following relations:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \operatorname{dom} f,\ \|x - y\| \geqslant (>, =)\varepsilon,\ \forall \lambda \in\,]0, 1[:\quad f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda)f(y) - \lambda(1-\lambda)\delta, \tag{2.5'}$$

or

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \operatorname{dom} f, \|x - y\| \geqslant (>, =)\varepsilon:\quad f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta. \tag{2.5''}$$

Remark 2.7. In [3] a function satisfying (2.5) is called strongly u.c., while f is called u.c. if δ: $R_+ \to R_+$ and $\delta(t_0) > 0$ for some $t_0 > 0$.

Vladimirov *et al.* [3] showed that the exact modulus of convexity μ satisfies the relation

$$\mu(ct) \geqslant c^2 \mu(t), \qquad \forall c \geqslant 1,\ t \geqslant 0. \tag{2.7}$$

Relation (2.7) says that the function $0 < t \to \mu(t)/t^2$ is nondecreasing. So, applying Proposition A.5, we can consider that the modulus of uniform convexity δ is l.s.c. and convex, $\delta(t) = 0 \Leftrightarrow t = 0$, $\operatorname{int}(\operatorname{dom} \delta) \neq \varnothing$ and $\liminf_{t\to\infty}(\delta(t)/t^2) > 0$. Denote by F_0 the class of such functions.

Remark 2.8. In Definition 2.2(i) we may take $\delta \in F_0$.

LEMMA 2.1. *If f is u.c. on* $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$, *then f is u.c. If* $\dim X < \infty$ *and f is u.c. on* $\operatorname{ri}(\operatorname{dom} f)$ (*relative interior*) *then f is u.c.*

Proof. By Remark 2.6

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in \operatorname{int}(\operatorname{dom} f), \ \|x - y\| \geqslant \varepsilon:$$

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta.$$

Let now $x, y \in \operatorname{dom} f \sim \operatorname{int}(\operatorname{dom} f)$, $\|x - y\| > \varepsilon$. Since $(x, f(x)) \in \operatorname{epi} f = \operatorname{cl}(\operatorname{int}(\operatorname{epi} f))$, there exists $((x_n, t_n)) \subset \operatorname{int}(\operatorname{epi} f)$, $(x_n, t_n) \to (x, f(x))$. So, $x_n \in \operatorname{int}(\operatorname{dom} f)$, $x_n \to x$, $f(x_n) < t_n \to f(x)$. Similarly, there exist $(y_n) \subset \operatorname{int}(\operatorname{dom} f)$, $(\tau_n) \subset R$ such that $y_n \to y$, $f(y_n) < \tau_n \to f(y)$. For n sufficiently large we have $\| x_n - y_n\| > \varepsilon$, so that

$$f\left(\frac{x_n + y_n}{2}\right) \leqslant \frac{1}{2} f(x_n) + \frac{1}{2} f(y_n) - \delta \leqslant \frac{1}{2} t_n + \frac{1}{2} \tau_n - \delta.$$

Passing to lower limit, knowing that f is l.s.c., we obtain

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta.$$

In a similar way we can treat the case $x \in \operatorname{dom} f \sim \operatorname{int}(\operatorname{dom} f)$ and $y \in \operatorname{int}(\operatorname{dom} f)$. So, f is u.c. The case $\dim X < \infty$ follows from the preceding case, replacing X by $X_0 = \operatorname{span}(\operatorname{dom} f - x_0)$ for some $x_0 \in \operatorname{dom} f$. ∎

THEOREM 2.2. *Let X be a reflexive Banach space. Then among the following conditions, the implications* (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) *are always valid*:

(i) *f is u.c.*,

(ii) $\exists \psi \in F_0$ (*or* F) $\forall x, y \in \operatorname{dom} f$: $f(y) \geqslant f(x) + f'(x; y - x) + \psi(\| y - x\|)$,

(iii) $\exists \psi \in F_0(F) \ \forall (x, x^*) \in \partial f, \ y \in \operatorname{dom} f$

$$f(y) \geqslant f(x) + \langle y - x, x^* \rangle + \psi(\| y - x\|),$$

(iv) $\exists \psi \in F_0(F) \ \forall (x, x^*) \in \partial f, \ y^* \in \operatorname{dom} f^*$:

$$f^*(y^*) \leqslant f^*(x^*) + \langle x, y^* - x^* \rangle + \psi^*(\| y^* - x^*\|),$$

(iii) *f^* is Fréchet equidifferentiable on* $\operatorname{int}(\operatorname{dom} f)$,

(vi) $\exists\psi \in F_0(F)\ \forall(x, x^*) \in \partial f,\ (y, y^*) \in \partial f$:

$$\langle x-y, x^*-y^*\rangle \geqslant \psi(\|x-y\|),$$

(vii) $\exists\varphi\colon R_+ \to R_+$ *nondecreasing, with* $\varphi_+(0) = 0$ *such that*

$$\|y-x\| \leqslant \varphi(\|y^*-x^*\|)\ \forall(x, x^*) \in \partial f,\ (y, y^*) \in \partial f,$$

(viii) ∂f^* *is single-valued and uniformly continuous on* $\mathrm{dom}(\partial f^*)$.

Moreover, if $\mathrm{int}(\mathrm{dom}\, f) \neq \varnothing$ *or* $\dim X < \infty$ *then all the above conditions are equivalent to*

(ix) $\exists\delta\colon R_+ \to R_+$ *with* $\delta(t) = 0 \Leftrightarrow t = 0$ *such that*

$$\forall x, y \in \mathrm{int}(\mathrm{dom}\, f)\ \exists x^* \in \partial f(x),\ y^* \in \partial f(y) \text{ with}$$

$$\langle y-x, y^*-x^*\rangle \geqslant \delta(\|y-x\|).$$

(*Take* r.i. *instead of* int *in the case* $\dim X < \infty$.)

Proof. For $\psi \in F$ in conditions (ii)–(iv), (vi), the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) follow from (or, as in) the preceding theorem. Let show that (ii) $\Rightarrow$ (i). Consider $x, y \in \mathrm{dom}\, f$; then

$$f(y) \geqslant f\left(\frac{x+y}{2}\right) + f'\left(\frac{x+y}{2}; \frac{y-x}{2}\right) + \psi\left(\frac{\|y-x\|}{2}\right),$$

$$f(x) \geqslant f\left(\frac{x+y}{2}\right) + f'\left(\frac{x+y}{2}; \frac{x-y}{2}\right) + \psi\left(\left\|\frac{x-y}{2}\right\|\right).$$

Adding these relations one obtains, taking into account that $f'((x+y)/2;\cdot)$ is sublinear,

$$f(x) + f(y) \geqslant 2f\left(\frac{x+y}{2}\right) + 2\psi\left(\left\|\frac{x-y}{2}\right\|\right).$$

Therefore condition (2.5″) is verified so that f is u.c. The implication (vii) $\Rightarrow$ (viii) is obvious, while for the converse implication take $\varphi\colon R_+ \to \bar{R}_+$,

$$\varphi(t) = \sup\{\|y-x\|\colon (x, x^*) \in \partial f,\ (y, y^*) \in \partial f,\ \|x^*-y^*\| \leqslant t\}. \quad (2.9)$$

It is clear that φ is nondecreasing, $\varphi_+(0) = 0$ and satisfies (vii). Moreover,

$$\limsup_{t\to\infty} \frac{\varphi(t)}{t} < \infty. \quad (2.10)$$

Firstly, note that by [12, Theorem 1], $\operatorname{dom}(\partial f^*) = X^*$. By the Corson–Klee lemma, since ∂f^* is uniformly continuous, ∂f^* is Lipschitz for large distances, i.e., there exists $L > 0$ such that, for $\|x^* - y^*\| \geqslant 1$,

$$\|\partial f^*(x^*) - \partial f^*(y^*)\| = \|x - y\| \leqslant L \|x^* - y^*\|,$$

so that for $t \geqslant 1$, $\varphi(t) \leqslant Lt$. Therefore (2.10) holds. So, in (iv) it can be taken $\psi^*(t) = \int_0^t \varphi(s)\, ds$. From (2.10) it follows that $\limsup_{t\to\infty} \psi^*(t)/t^2 < \infty$ (for $t \geqslant 1$, $\psi^*(t) = \psi^*(1) + \int_1^t \varphi(s)\, ds \leqslant \psi^*(1) + L(t^2 - 1)/2$, so that $\limsup_{t\to\infty} \psi^*(t)/t^2 \leqslant L/2 < \infty$). Applying Proposition A.4(i), $\liminf_{t\to\infty} \psi(t)/t^2 > 0$, i.e., $\psi \in F_0$.

So if any of the conditions (iii), (iv), (vi) is satisfied with $\psi \in F$, by the above reasoning we can find $\psi \in F_0$ such that the above conditions hold. In condition (ii) we can take $\psi \in F_0$ from (i) $\Leftrightarrow$ (ii) (with $\psi \in F$) and Remark 2.8.

If $\operatorname{int}(\operatorname{dom} f) \neq \varnothing$, the implication (iii) $\Rightarrow$ (i) follows from (1.4) and Lemma 2.1.

It is clear that (vi) $\Rightarrow$ (ix). The implication (ix) $\Rightarrow$ (i) follows from [3, Theorem 2]: *Let $f\colon U \to R$, $U \subset X$ a convex set and f convex such that $\partial f(x) \neq \varnothing$ $\forall x \in U$. If*

$$\forall x, y \in U\ \exists x^* \in \partial f(x),\ y^* \in \partial f(y),\ \langle x - y, x^* - y^* \rangle \geqslant \xi(\|x - y\|),$$

where ξ is a nonnegative and measurable function, positive on a set of positive measure than f is u.c. (see Remark 2.7) *with $\delta(t) = \int_0^t (\xi(s)/s)\, ds$,* also using Lemma 2.1. ∎

COROLLARY 2.4. *If X is reflexive and f satisfies some of the conditions* (i)–(viii), *then* $\operatorname{dom} f^* = X^*$ *and* $\lim_{\|x\|\to\infty} (f(x)/\|x\|) = \infty$ (*in fact* $\liminf_{\|x\|\to\infty} (f(x)/\|x\|^2) > 0$).

Proof. If one of the conditions (i)–(viii) is valid, then (viii) is true so that $X^* = R(\partial f) = \operatorname{dom} f^*$, and (iii) is valid with $\psi \in F_0$, too. Taking $(\bar{x}, \bar{x}^*) \in \partial f \neq \varnothing$, we have

$$f(x) \geqslant f(\bar{x}) + \langle x - \bar{x}, \bar{x}^* \rangle + \psi(\|x - \bar{x}\|), \qquad \forall x \in X.$$

Dividing by $\|x\|^2$ and taking the lower limit, we obtain

$$\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} \geqslant \liminf_{t\to\infty} \frac{\psi(t)}{t^2} > 0. \quad ∎$$

PROPOSITION 2.1. *Suppose condition* (iii) *of Theorem* 2.2 *holds with $\psi \in F_0$ for which the map $x \to \psi(\|x\|)$ is u.c. Then f is u.c.*

The proof is similar to that of [2, Proposition 6, (c) $\Rightarrow$ (b)].

Remark 2.9. Theorem 2.2 generalizes [1, Lemma, p. 288]. Taking into account Remark 4.2, Theorem 2.2 also generalizes [2, Proposition 6].

Remark 2.10. The implication (i) $\Rightarrow$ (ii) is proved in [7, Lemma II.1.2], the converse one being proved in [7] for $\psi(t) = at^2$, since a function $\gamma: R_+ \to R_+$ satisfies $\gamma(ct) \geqslant c\gamma(t)$, $\forall c, t \geqslant 0$ iff $\gamma(t) = at$ for some $\alpha \geqslant 0$.

PROPOSITION 2.2. *Let X be reflexive and $\lim_{\|x\| \to \infty}(f(x)/\|x\|) = \infty$. If f is u.c. on every bounded subset of its domain, then f if Fréchet differentiable and $\partial f^* = \nabla f^*$ is uniformly continuous on bounded sets of X^*.*

Proof. Since f is u.c. on bounded subsets of its domain, it follows that f is u.c. at any $x \in \operatorname{dom} f$, so that, by Corollary 2.2, f^* is Fréchet differentiable on $\operatorname{int}(\operatorname{dom} f^*)$. But, from $\lim_{\|x\| \to \infty}(f(x)/\|x\|) = \infty$, $R(\partial f) = X^* = \operatorname{dom} f^*$ and ∂f^* maps bounded sets into bounded sets. So, let $B \subset X^*$ be a bounded set; then for some $M > 0$, $B \subset \partial f(\bar{S}(0, M))$. The function $f + I_{\bar{S}(0,M)}$ is u.c. so that, by Theorem 2.2, $(\partial(f + I_{\bar{S}(0,M)}))^{-1}$ is uniformly continuous. But $(\partial(f + I_{\bar{S}(0,M)}))^{-1}|_B = (\partial f)^{-1}|_B$ so that $(\partial f)^{-1} = \nabla f^*$ is uniformly continuous on B. ∎

3. THE CASE $X = R$

In this section, $f: R \to R \cup \{\infty\}$ will be a proper l.s.c. convex function with $\operatorname{int}(\operatorname{dom} f) \neq \varnothing$.

PROPOSITION 3.1. *f is u.c. at $\bar{x} \in \operatorname{dom}(f)$ if and only if*

$$x \in \operatorname{dom} f,\ x > \bar{x} \Rightarrow f'_+(x) > f'_+(\bar{x})$$

and (3.1)

$$x \in \operatorname{dom} f,\ x < \bar{x} \Rightarrow f'_-(x) < f'_-(\bar{x}).$$

If $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$ then the conditions (i)–(viii) *of Theorem* 2.1 *are actually equivalent to* (3.1).

Proof. Consider the case $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$. Taking into account Proposition A.1(iv), it is clear that (vii) from Theorem 2.1 implies (3.1) so that the necessity follows. Suppose that f is not u.c. at $\bar{x}$. Then $\exists \varepsilon > 0$ $\forall n \in N^*$ $\exists x_n \in \operatorname{dom}(f)$, $|x_n - \bar{x}| = \varepsilon$ such that

$$f\left(\frac{1}{2}x_n + \frac{1}{2}\bar{x}\right) > \frac{1}{2}f(\bar{x}) + \frac{1}{2}f(x_n) - \frac{1}{n}.$$

We can consider $x_n = \bar{x} + \varepsilon$. The above inequality shows that

$$f\left(\frac{1}{2}(\bar{x}+\varepsilon)+\frac{1}{2}\bar{x}\right)=\frac{1}{2}f(\bar{x}+\varepsilon)+\frac{1}{2}f(\bar{x}),$$

so that, by Proposition A.1(iii), f is affine on $[\bar{x}, \bar{x}+\varepsilon]$; therefore $f'_-(\bar{x}+\varepsilon) = f'_+(\bar{x}) = f'_+(\bar{x}+\varepsilon/2)$, i.e., (3.1) is not true. Hence (3.1) $\Rightarrow f$ is u.c. at $\bar{x}$. The remaining case can be proved similarly. ∎

THEOREM 3.1. *The following conditions are equivalent*:

(i) f *is u.c. on every interval* $[a, b]$, $[a, b] \cap \operatorname{int}(\operatorname{dom} f) \neq \varnothing$,

(ii) f *is u.c. at any* $x \in \operatorname{dom} f$,

(iii) $f'_+ (\Leftrightarrow f'_-)$ *is increasing on* $\operatorname{dom} f$,

(iv) f *is strictly convex.*

Proof. (i) $\Rightarrow$ (ii) is obvious, the implication (ii) $\Rightarrow$ (iii) follows from the preceding proposition, and (iii) $\Leftrightarrow$ (iv) is stated by Proposition A.1(ii). So, let us show (iii) $\Rightarrow$ (i). Suppose there exist $a, b \in R$, $a < b$, $[a, b] \cap \operatorname{int}(\operatorname{dom} f) \neq \varnothing$ such that f is not u.c. on $[a, b]$. Hence, $\exists \varepsilon > 0$ $\forall n \in N^*$ $\exists x_n, y_n \in [a, b] \cap \operatorname{dom} f$, $|x_n - y_n| = 4\varepsilon$ such that

$$\frac{1}{n} > \frac{1}{2}f(x_n) + \frac{1}{2}f(y_n) - f\left(\frac{x_n+y_n}{2}\right).$$

We may take $y_n = x_n + 4\varepsilon$. The above relation becomes, taking into account Proposition A.1(vi),

$$\begin{aligned}\frac{1}{n} &> \frac{1}{2}f(x_n) + \frac{1}{2}f(x_n+4\varepsilon) - f(x_n+2\varepsilon)\\ &\geqslant \frac{1}{2}f(x_n) + \frac{1}{2}f(x_n+2\varepsilon) - f(x_n+\varepsilon).\end{aligned} \tag{3.2}$$

Since (x_n) is bounded, we can suppose $x_n \to x \in \operatorname{cl}(\operatorname{dom} f)$. Since $f|\operatorname{cl}(\operatorname{dom} f)$ is continuous, $f(x_n) \to f(x)$. On the other hand, since x_n, $x_n + 4\varepsilon \in \operatorname{dom} f$, it follows that x, $x + 4\varepsilon \in \operatorname{cl}(\operatorname{dom} f) = \operatorname{cl}(\operatorname{int}(\operatorname{dom} f))$, so that $x+\varepsilon$, $x + 2\varepsilon \in \operatorname{int}(\operatorname{dom} f)$. Taking the limit in (3.2) we obtain

$$0 \geqslant \frac{1}{2}f(x) + \frac{1}{2}f(x+2\varepsilon) - f(x+\varepsilon) \geqslant 0.$$

Thus f is affine on $[x, x+2\varepsilon]$, so that f'_+ is not increasing, a contradiction. ∎

Remark 3.1. If f is u.c. then $f'_+(f'_-)$ is increasing on dom f, and

$$\liminf_{x\to\infty}\frac{f'_+(x)}{x}>0, \qquad \liminf_{x\to-\infty}\frac{f'_+(x)}{x}>0. \tag{3.3}$$

Indeed, by Theorem 3.1, f'_+ is increasing, and taking into account Proposition A.1(iv) and Theorem 2.2, for some $\psi\in F_0$ and $\bar{x}\in \operatorname{int}(\operatorname{dom} f)$,

$$(x-\bar{x})(f'_+(x)-f'_+(\bar{x}))\geqslant\psi(|x-\bar{x}|),$$

so that

$$\frac{f'_+(x)-f'_+(\bar{x})}{x-\bar{x}}\geqslant\frac{\psi(|x-\bar{x}|)}{|x-\bar{x}|^2},$$

from which (3.3) follows.

Let us show that (3.3) and f'_+ increasing on dom f do not assure that f is u.c.

EXAMPLE. Let $f\colon R\to\bar{R}_+$ be the l.s.c. convex function for which f'_+ is defined by

$$\begin{aligned} f'_+(x)&=-\infty \qquad \text{for} \quad x<0,\\ &=n+\frac{1}{n+1}(x-n) \qquad \text{for} \quad x\in[n,n+1[,\ n\in N. \end{aligned}$$

It is clear that f'_+ is increasing on $\operatorname{dom} f=R_+$ and $\liminf_{x\to\infty}(f'_+(x)/x)=1>0$ and $\liminf_{x\to-\infty}(f'_+(x)/x)=\infty>0$, f is not u.c. since condition (vi) of Theorem 2.2 is not verified. Indeed, $(x_n,x_n^*)=(n,n)\in\partial f$, $(y_n,y_n^*)=(n+\frac{1}{2},n+(1/2(n+1)))\in\partial f$, $|x_n-y_n|=\frac{1}{2}$, but $\langle x_n-y_n,x_n^*-y_n^*\rangle=1/(4(n+1))\to 0$.

An important class of convex functions on R is $\{f_p\colon p\in[1,\infty[\}$, where $f_p(x)=|x|^p$. For $p=1$, f_1 is not strictly convex. For $p\in]1,2]$ f_p is strictly convex, and so, by Theorem 3.1, f_p is u.c. on every bounded interval, but f_p is not u.c.; indeed $f'_p(x)=p\,|x|^{p-2}x$ so that (3.3) is not verified.

For $p\geqslant 2$, the following result is true.

PROPOSITION 3.2. *If $p\in[2,\infty[$ then f_p is u.c. The exact modulus of convexity μ of f_p satisfies*

$$p^{-1}\cdot 2^{-p(p-2)/(p-1)}\cdot t^p\leqslant\mu(t)\leqslant t^p, \qquad \forall t\in R_+. \tag{3.4}$$

Proof. f_p is differentiable and $f_p(x) = p\,|x|^{p-2} \cdot x$. From Theorem 2.2 we have

$$f_p(x) \geqslant f_p(0) + f_p'(0; x) + \mu(|x|), \qquad \forall x \in R,$$

so that $\mu(t) \leqslant t^p$ $\forall t \in R_+$. The rest of the proof follows from Theorem 2.2, taking into account the relations between the functions ψ and φ which appear in the proof of the theorem, and in the (well-known) lemma below.

LEMMA 3.1. *Let $p \in [2, \infty[$. Then for every $x, y \in R$*

$$(|x|^{p-2} \cdot x - |y|^{p-2} \cdot y)(x - y) \geqslant 2^{2-p}\,|x - y|^p.$$

The fact that f_p, $p \geqslant 2$, is u.c. follows from a result of [3] (see Remark 4.2), too.

4. EXAMPLES

Let X be a Banach space, $\varphi: R_+ \to \bar{R}_+$ a nondecreasing function with $\operatorname{int}(\operatorname{dom}\varphi) \neq \varnothing$, to which we associate ψ, f and J_ω as in the Appendix. Let $0 < a = \sup(\operatorname{dom}\varphi)$.

THEOREM 4.1. (i) *f is uniformly convex at any $x \in \operatorname{int}(\operatorname{dom} f) = S(0, a)$ iff φ is increasing on* $\operatorname{dom}\varphi$ *and X is locally uniformly convex.*

(ii) *f is u.c. on every $S(0, M)$, $M \in]0, a[$ iff φ is increasing on* $\operatorname{dom}\varphi$ *and X is uniformly convex.*

Proof. Let f be u.c. on $S(0, M)$; then it follows that f is u.c. on the set $\{tx: t \in]0, M[\}$ where $x \in S_X = S$. This one is nothing else but ψ is u.c. on $]0, M[$. Applying Theorem 3.1 φ is increasing on $\operatorname{dom}\varphi$. Fix now $t \in]0, a[$ and take $\varepsilon > 0$. Then for every $x, y \in S$ with $\|x - y\| \geqslant \varepsilon$ we have

$$\psi\left(\frac{t}{2}\|x + y\|\right) \leqslant \frac{1}{2}\psi(t) + \frac{1}{2}\psi(t) - \frac{1}{4}\delta(t\varepsilon)$$
$$= \psi(t) - \frac{1}{4}\delta(t\varepsilon) = \psi(t - t\delta'),$$

for some $\delta' > 0$, since ψ is continuous on $[0, a[$. Since ψ is increasing from the above relation it follows that $\|x + y\| \leqslant 2(1 - \delta')$, therefore X is uniformly convex.

In a similar way, from the uniform convexity of f at any $x \in S(0, a)$, it follows that φ is increasing on $\operatorname{dom}\varphi$ and X is locally uniformly convex.

To prove the sufficiency in (i) and (ii) we shall need some lemmas. Fix $M \in]0, a[$ and $\varepsilon > 0$ and take

$$E = E(\varepsilon, M) = \{(x, y) \in X \times X : x, y \in S(0, M, \|x - y\| \geqslant \varepsilon\}.$$

W.l.o.g. we can suppose $\varepsilon < M$, so that $E \neq \emptyset$.

In the following lemmas φ is increasing.

LEMMA 4.1.

$$\exists \eta > 0,\ \delta > 0\ \forall (x, y) \in E, \|x\| \leqslant \eta:$$

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta.$$

Proof. Suppose the assertion is not true. Then $\forall n \in N^*\ \exists (x_n, y_n) \in E$, $x_n \to 0$ such that

$$\frac{1}{n} > \frac{1}{2} f(x_n) + \frac{1}{2} f(y_n) - f\left(\frac{x_n + y_n}{2}\right)$$

$$\Leftrightarrow \psi\left(\left\|\frac{x_n + y_n}{2}\right\|\right) + \frac{1}{n} > \frac{1}{2}\psi(\|x_n\|) + \frac{1}{2}\psi(\|y_n\|).$$

Let $t = \limsup_{n\to\infty} \|y_n\| \leqslant M < a$; then $\limsup \|(x_n + y_n)/2\| = t/2$. Since ψ is increasing and continuous on $[0, a[$, taking the upper limit in the above inequality, we obtain $\psi(t/2) \geqslant \frac{1}{2}\psi(t)$. If $t > 0$, by the strict convexity of ψ, we have $\psi(t/2) < \frac{1}{2}\psi(t)$, a contradiction, so that $t = 0$. Therefore $y_n \to 0$, which together with $x_n \to 0$ contradicts $(x_n, y_n) \in E$ for every $n \in N^*$. ∎

LEMMA 4.2. *Let* $0 < m < M$. *Then* $\exists \eta, \mu, \delta > 0\ \forall \alpha, \beta \in [m, M]$ *with* $|\alpha - \beta| < \eta$, $\forall x, y \in S$ *with* $\|x + y\| \leqslant \mu$:

$$\psi\left(\frac{\|\alpha x + \beta y\|}{2}\right) \leqslant \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta) - \delta.$$

Proof. Suppose the assertion of the lemma does not hold. Then $\forall n \in N^*$ $\exists \alpha_n, \beta_n \in [m, M]$ with $\alpha_n - \beta_n \to 0$, $\exists x_n, y_n \in S$ with $x_n + y_n \to 0$ such that

$$\psi\left(\frac{\|\alpha_n x_n + \beta_n y_n\|}{2}\right) > \frac{1}{2}\psi(\alpha_n) + \frac{1}{2}\psi(\beta_n) - \frac{1}{n}.$$

We can consider that $\alpha_n, \beta_n \to \alpha \in [m, M]$. Then $(\alpha_n x_n + \beta_n y_n)/2 \to 0$. Taking the limit in the above inequality we obtain $0 = \psi(0) \geqslant \psi(\alpha)$ so that $\alpha = 0$, a contradiction. ∎

LEMMA 4.3. *Let* $0 < m < a$, $2 > \mu > 0$. *Then* $\exists \delta > 0$ $\forall \alpha, \beta \in [m, \infty[\cap \operatorname{dom} \psi$, $\forall x, y \in S$ *with* $\|x + y\| \geqslant \mu$:

$$\psi\left(\left\|\frac{\alpha x + \beta y}{2}\right\|\right) \leqslant \frac{1}{2}\psi(\beta) + \frac{1}{2}\psi(\alpha) - \delta\left(1 - \left\|\frac{x + y}{2}\right\|\right).$$

Proof. Take $\alpha, \beta \in [m, \infty[\cap \operatorname{dom} \psi$ and $x, y \in S$, $\|x + y\| \geqslant \mu$; suppose $\alpha \leqslant \beta$. Then

$$\left\|\frac{\alpha x + \beta y}{2}\right\| \leqslant \frac{\alpha}{2}\|x + y\| + \frac{\beta - \alpha}{2} = \gamma.$$

ψ being increasing, we have

$$\begin{aligned}
\frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta) - \psi\left(\frac{\|\alpha x + \beta y\|}{2}\right) &\geqslant \frac{1}{2}\psi(\alpha) + \frac{1}{2}\psi(\beta) - \psi(\gamma) \\
&= \frac{1}{2}(\psi(\alpha) - \psi(\gamma)) + \frac{1}{2}(\psi(\beta) - \psi(\gamma)) \\
&\geqslant \frac{1}{2}\varphi_-(\gamma)(\alpha - \gamma) + \frac{1}{2}\varphi_-(\gamma)(\beta - \gamma) = \frac{1}{2}\varphi_-(\gamma)(\alpha + \beta - 2\gamma) \\
&= \alpha\varphi_-(\gamma)\left(1 - \left\|\frac{x + y}{2}\right\|\right) \geqslant m\varphi_-\left(\frac{m\mu}{2}\right)\left(1 - \left\|\frac{x + y}{2}\right\|\right).
\end{aligned}$$

Taking $\delta = m\varphi_-(m\mu/2) > 0$, the proof is complete. ∎

Proof of Theorem 4.1 (*continued*). Let $0 < M < a$ and $\varepsilon > 0$ ($\varepsilon < M$) be fixed. Let $\eta_1, \delta_1 > 0$ given by Lemma 4.1; we may suppose $0 < \eta_1 < M$. Take η_2, μ_2, $\delta_2 > 0$ given by Lemma 4.2 with $m = \eta_1$. W.l.o.g. we suppose $0 < \mu_2 < 2$, $0 < \eta_2 < \varepsilon/2$. Let $\delta_3 > 0$ be given by Lemma 4.3 with $m = \eta_1$, $\mu = \mu_2$.

Let now $x, y \in S(0, M)$, $\|x - y\| \geqslant \varepsilon$. For such x, y we have at least one of the following possibilities:

(a) $\|x\| \leqslant \eta_1$ or $\|y\| \leqslant \eta_1$,

(b) $|\|x - \|y\|| \geqslant \eta_2$,

(c) $\|x\|, \|y\| \geqslant \eta_1$, $|\|x\| - \|y\|| \leqslant \eta_2$, $\|x/\|x\| + y/\|y\|\| \leqslant \mu_2$,

(d) $\|x\|, \|y\| \geqslant \eta_1$, $|\|x\| - \|y\|| \leqslant \eta_2$, $\|x/\|x\| + y/\|y\|\| \geqslant \mu_2$.

(a) By way of choosing η_1 and δ_1, for $\delta = \delta_1$, we have

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \delta. \tag{4.1}$$

(b) Since φ is increasing and $M < a$, by Theorem 3.1, ψ is u.c. on $[0, M[$. So, for $\eta_2 > 0$ there exists $\delta' > 0$ such that (2.5″) holds on $[0, M[$. Therefore

$$\psi\left(\left\|\frac{x+y}{2}\right\|\right) \leqslant \psi\left(\frac{\|x\| + \|y\|}{2}\right) \leqslant \frac{1}{2}\psi(\|x\|) + \frac{1}{2}\psi(\|y\|) - \delta',$$

so that (4.1) holds with $\delta = \delta'$.

(c) Let $\alpha = \|x\|$, $\beta = \|y\|$; hence $\alpha, \beta \in [\eta_1, M]$, $|\alpha - \beta| \leqslant \eta_2$. Let $u = x/\|x\|$, $v = y/\|y\| \in S$; moreover, $\|u + v\| \leqslant \mu_2$. The way of choosing η_2, μ_2, δ_2 shows that (4.1) holds with $\delta = \delta_2$.

(d) Let X be uniformly convex. With the notations from case (c) we have $\alpha, \beta \in [\eta_1, M] \subset \operatorname{dom}\psi$, $u, v \in S$, $\|u + v\| \geqslant \mu_2$, $|\alpha - \beta| \leqslant \eta_2 \leqslant \varepsilon/2$. So,

$$\begin{aligned}\|u - v\| &= \left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| = \frac{1}{\beta}\left\|(x - y) + \frac{\beta - \alpha}{\alpha} x\right\| \\ &\geqslant \frac{1}{\beta}(\|x - y\| - |\beta - \alpha|) \geqslant \frac{\varepsilon}{2M} = \varepsilon''.\end{aligned}$$

For $\varepsilon'' > 0$ (depending only of ε and M), since X is uniformly convex, there exists $\delta'' > 0$ such that $1 - \|(u + v)/2\| \geqslant \delta''$. So that, by way of choosing δ_3, (4.1) holds with $\delta = \delta_3\delta''$. Taking $\delta = \min\{\delta_1, \delta', \delta_2, \delta_3 \cdot \delta''\}$, (4.1) holds for any $x, y \in S(0, M)$ with $\|x - y\| \geqslant \varepsilon$. Therefore f is u.c. on $S(0, M)$.

If X is locally uniformly convex, fix $\bar{x} \in \operatorname{int}(\operatorname{dom} f) = S(0, a)$ and $M < a$ such that $\|\bar{x}\| < M$. Then, as above, $\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in S(0, M)$

$$f\left(\frac{\bar{x}+y}{2}\right) \leqslant \frac{1}{2} f(\bar{x}) + \frac{1}{2} f(y) - \delta.$$

Using Proposition A.1(vi) it follows that the above inequality holds for any $y \in \operatorname{dom} f$. ∎

Remark 4.1. In [8] it is shown that $f: X \to R$, $f(x) = \|x\|^2$ has the property $\inf\{f(x) + f(y) - 2f((x + y)/2)\colon\ y \in S,\ \|x - y\| \geqslant \varepsilon\} > 0\ \ \forall \varepsilon > 0$ iff X is uniformly convex, and $\inf\{f(x) + f(y) + 2f((x + y)/2)\colon \|x - y\| \geqslant \varepsilon\} > 0\ \forall \varepsilon > 0\ \forall x \in X$ iff X is locally uniformly convex.

Remark 4.2. Vladimirov *et al.* [3] showed that if X is a Hilbert space and $\varphi: R_+ \to R_+$ has the property $\varphi(ct) \geqslant c\varphi(t)\ \ \forall c \geqslant 1,\ t \geqslant 0$, then the function $f: X \to R_+$, $f(x) = \int_0^{\|x\|} \varphi(t)\, dt$ is uniformly convex with modulus of convexity $\delta(t) = \int_0^t \varphi(s/2)\, ds$.

Remark 4.3. Taking $\psi(t) = t^p$, $p > 1$ and X uniformly convex in Theorem 4.1, we obtain f is u.c. on every bounded convex set, just the statement of Theorem 6 in [3] (in fact, this result is also proved in [11, p. 54]).

COROLLARY 4.1. *Let $\psi: R_+ \to \bar{R}_+$ be increasing and X be locally uniformly convex and reflexive. Then $(J_\varphi)^{-1}$ is single-valued and continuous on the set $S(0, M) \subset X^*$, where $M = \varphi_-(a)$.*

Proof. By Theorem 4.1 f is u.c. at any $x \in S(0, a)$ so that, applying Corollary 2.2, $(\partial f)^{-1} = (J_\varphi)^{-1}$ is single-valued and continuous on $J_\varphi(S(0, a)) = S(0, M)$. ∎

COROLLARY 4.2. *Let $\varphi: R_+ \to R_+$ be increasing with $\lim_{t\to\infty} \varphi(t) = \infty$. Then $(J_\varphi)^{-1}$ is single-valued and uniformly continuous on bounded sets if and only if X is uniformly convex.*

Proof. The conditions upon φ show that $\operatorname{dom} f = X$ and $\lim_{\|x\|\to\infty} f(x)/\|x\| = \infty$. If X is uniformly convex, by Theorem 4.1, f is u.c. on bounded sets, so that, applying Proposition 2.2, $(J_\varphi)^{-1} = (\partial f)^{-1}$ is single-valued and uniformly continuous on bounded subsets of X^*. For the converse implication, the proof is sketched. Take $\bar{S}(0, M) \subset X$; $(J_\varphi)^{-1}$ is uniformly continuous on $\bar{S}(0, 2\varphi_+(M)) \subset X^*$. So, there exists a nondecreasing function φ_0 with $\varphi_{0+}(0) = 0$ such that $\|J_\varphi^{-1}(x^*) - J_\varphi^{-1}(y^*)\| \leqslant \varphi_0(\|x^* - y^*\|)$. Take $\varphi_1 = \varphi_0 + I_{[0, \varphi_+(M)]}$; then φ_1 is nondecreasing and $\varphi_{1+}(0) = 0$. We have then

$$\|J_\varphi^{-1}(x^*) - J_\varphi^{-1}(y^*)\| \leqslant \varphi_1(\|x^* - y^*\|), \qquad \forall x^* \in \bar{S}(0, \varphi_+(M)),\ y^* \in X^*.$$

As in the proof of Theorem 2.1, there exists $\psi_1 \in F$ such that

$$f(y) \geqslant f(x) + \langle y - x, x^* \rangle + \psi_1(\|y - x\|),$$
$$\forall x \in \bar{S}(0, M),\ y \in X,\ x^* \in J_\varphi(x),$$

so that

$$f(y) \geqslant f(x) + f'(x; y - x) + \psi_1(\|y - x\|), \qquad \forall x \in \bar{S}(0, M),\ y \in X.$$

Let $x, y \in \bar{S}(0, M)$. Then $(x + y)/2 \in \bar{S}(0, M)$, so that, as in the proof of Theorem 2.2 (ii) $\Rightarrow$ (i),

$$f\left(\frac{x + y}{2}\right) \leqslant \frac{1}{2} f(x) + \frac{1}{2} f(y) - \psi_1\left(\frac{\|y - x\|}{2}\right),$$

so that f is u.c. on $\bar{S}(0, M)$. Applying Theorem 4.1, X is uniformly convex. ∎

Remark 4.4. Corollary 4.2 generalizes Theorem 2.10 in [8].

Remark 4.5. A similar result to that in Corollary 4.2 can be stated even for $\varphi: R_+ \to \bar{R}_+$ increasing on $\operatorname{dom} \varphi$: X is uniformly convex iff $J_\varphi^{-1} \mid S(0, \varphi_-(a))$ is single-valued and J_φ^{-1} is uniformly continuous on every $S(0, M)$, $M < \varphi_-(a)$.

An important class of convex functions is the class of convex integrands. Let Ω denote a measure space with a finite measure ν and let $L^p(\Omega, R^n)$ be the space consisting of all R^n valued measurable functions u on Ω such that $\int_\Omega \|u(x)\|^p \, d\nu(x) < \infty$, $1 \leqslant p < \infty$, the norm on R^n being the Euclidean norm. To establish a criterion for a convex integrand be u.c. we need the following.

LEMMA 4.4. *Suppose that* $f: R_+ \to \bar{R}_+$ *is nondecreasing with* $f(t) = 0 \Leftrightarrow t = 0$, *and* $g: L^p(\Omega) = L^p(\Omega, R^1) \to \bar{R}_+$, $g(u) = \int_\Omega f(|u(x)|) \, d\nu(x)$ $(1 \leqslant p < \infty)$. *If*

(i) $\liminf_{x \to \infty} f(x)/x^p > 0$, *then*

(ii) $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall u \in L^p(\Omega)$, $\|u\| \geqslant \varepsilon \Rightarrow g(u) \geqslant \delta$.

If the measure ν *is continuous* (*i.e.*, $\forall \lambda \in \,]0, \nu(\Omega)[\; \exists A \subset \Omega$, *$A$ measurable, such that* $\nu(A) = \lambda$), *then* (ii) $\Rightarrow$ (i).

Proof. W.l.o.g, we can take $p = 1$ (otherwise take $\bar{f}: R_+ \to \bar{R}_+$, $\bar{f}(x) = f(x^{1/p})$; it is clear that $\bar{f}(x) = 0 \Leftrightarrow x = 0$, $\bar{f}$ is nondecreasing and $\liminf_{x \to \infty}(\bar{f}(x)/x) = \liminf_{x \to \infty}(f(x)/x^p)$, $g(u) = \int_\Omega f(|u(x)|) \, d\nu(x) = \int_\Omega \bar{f}(|u(x)|^p) \, d\nu(x)$).

(i) $\Rightarrow$ (ii). Let $\varepsilon > 0$ and put $\alpha = \varepsilon/(2\nu(\Omega))$; since $\liminf_{x \to \infty} f(x)/x > 0$, there exist $\beta > 0$ such that $f(x) \geqslant \beta x \; \forall x \geqslant \alpha$. Let $u \in L^1(\Omega)$, $\|u\| \geqslant \varepsilon$ and put $A = \{x \in \Omega : |u(x)| < \alpha\}$. Then $\int_A |u(x)| \, d\nu(x) \leqslant \varepsilon/2$, so that

$$\begin{aligned} \int_{\Omega \sim A} |u(x)| \, d\nu(x) &= \int_\Omega |u(x)| \, d\nu(x) - \int_A |u(x)| \, d\nu(x) \\ &\geqslant \varepsilon - \varepsilon/2 = \varepsilon/2. \end{aligned}$$

Therefore

$$\begin{aligned} g(u) = \int_\Omega f(|u(x)|) \, d\nu(x) &\geqslant \int_{\Omega \sim A} f(|u(x)|) \, d\nu(x) \\ &\geqslant \int_{\Omega \sim A} \beta \, |u(x)| \, d\nu(x) \geqslant \frac{\beta\varepsilon}{2} = \delta > 0. \end{aligned}$$

Suppose ν is continuous and (i) is not true, i.e., $\liminf_{x\to\infty}(f(x)/x)=0$. Then there exists a sequence $(x_n)\subset R_+$, $x_n\to\infty$ such that $f(x_n)/x_n\to 0$. Since $1/x_n\to 0$ we can suppose $1/x_n<\nu(\Omega)$ $\forall n\in N$. Let $A_n\subset\Omega$ be measurable such that $\nu(A_n)=1/x_n$, and let $u_n:\Omega\to R$, $u_n(x)=x_n$ for $x\in A_n$, $u_n(x)=0$ for $x\in\Omega\sim A_n$. Then $\int_\Omega |u_n(x)|\,d\nu(x)=1$ and $\int_\Omega |f(|u_n(x)|)\,d\nu(x)=f(x_n)/x_n\to 0$, so that (ii) is not valid. So, (ii) $\Rightarrow$ (i) and the lemma is proved. ∎

Remark 4.6. If $\nu(\Omega)=\infty$ the implication (i) $\Rightarrow$ (ii) in Lemma 4.1 is not generally true. For example, take $\Omega=R_+$, ν the Lebesgue measure on Ω, $f(x)=\min\{x,x^2\}$ and $u_n(x)=0$ for $x<1$ and $u_n(x)=1/(n\cdot x^{1+1/n})$ for $x\geqslant 1$. Then $\|u_n\|_1=1$ and $\|f(|u_n|)\|_1=1/n(n+2)\to 0$.

THEOREM 4.2. *Let $f:R^n\to R\cup\{\infty\}$ be a u.c. function with modulus of convexity $\mu(:R_+\to\bar{R}_+)$ and Ω a measure space with finite measure ν. If* $\liminf_{t\to\infty}(\mu(t)/t^p)>0$ *then the function $g:L^p(\Omega,R^n)\to R\cup\{\infty\}$,*

$$\begin{aligned} g(u)&=\int_\Omega f(u(x)\,d\nu(x) \qquad \textit{if} \quad f(u)\in L^1(\Omega),\\ &=+\infty \qquad \text{otherwise,}\end{aligned} \tag{4.2}$$

is uniformly convex.

Proof. It is known (see [13]) that g is a proper l.s.c. convex function. Let $\varepsilon>0$ and $u,v\in\operatorname{dom} g\subset L^p(\Omega,R^n)$, $w=u-v$, $\|w\|_p\geqslant\varepsilon$. Since f is u.c. we have

$$f\left(\frac{u(x)+v(x)}{2}\right)\leqslant\frac{1}{2}f(u(x))+\frac{1}{2}f(v(x))-\frac{1}{4}\mu(\|w(x)\|).$$

Integrating over Ω we obtain

$$g\left(\frac{u+v}{2}\right)\leqslant\frac{1}{2}g(u)+\frac{1}{2}g(v)-\frac{1}{4}\int_\Omega\mu(\|w(x)\|)\,d\nu(x).$$

From the above lemma, it follows there exists $\delta>0$ (depending only of ε) such that $\int_\Omega\mu(\|w(x)\|)\,d\nu(x)>4\delta$. Therefore

$$\forall u,v\in\operatorname{dom} g,\ \|u-v\|_p\geqslant\varepsilon:\ g\left(\frac{u+v}{2}\right)\leqslant\frac{1}{2}g(u)+\frac{1}{2}g(v)-\delta,$$

so that g is u.c. ∎

Remark 4.7. The result of Theorem 4.2 remains true if we replace R^n by a Banach space X.

COROLLARY 4.3. *If* $f: R^n \to R \cup \{\infty\}$ *is u.c. and* $1 \leqslant p \leqslant 2$ *then* g *defined by* (4.2) *is u.c.* ($\nu(\Omega) < \infty$).

Proof. Let μ be the exact modulus of convexity of f. Then, from (2.7), $\liminf_{t\to\infty}(\mu(t)/t^2) > 0$, so that $\liminf_{t\to\infty}(\mu(t)/t^p) > 0$ for $p \in [1, 2]$. Applying Theorem 4.2 the statement follows. ∎

COROLLARY 4.4. *Let* $g: L^p(\Omega, R^n) \to R \cup \{\infty\}$, $g(u) = \int_\Omega \|u(x)\|^q \, d\nu(x)$ ($\nu(\Omega) < \infty$). *If* $q \geqslant \max\{p, 2\}$ *then* g *is u.c.* (*R^n being endowed with the Euclidean norm*).

Proof. By Remark 4.2, the function $f: R^n \to \bar{R}_+$, $f(x) = \|x\|^q$ is u.c. for $q \geqslant 2$ with $\delta(t) = \int_0^t q \cdot (s/2)^{q-1} \, ds = t^q/2^{q-1}$. Therefore $\liminf_{t\to\infty}(\delta(t)/t^p > 0$ for $q \geqslant p$, so that, applying the preceding theorem, g is u.c. ∎

Remark 4.8. If the condition $\liminf_{t\to\infty}(\delta(t)/t^p) > 0$ is dropped, g can be not u.c. as shown by the following example.

Let $g: L^3(0, 1) \to R$, $g(u) = \int_0^1 |u(x)|^2 \, dx$. g is not u.c., otherwise for $\varepsilon = 1$ there exists $\delta > 0$ such that for $u, v \in L^3(0, 1)$, $\|u - v\|_3 \geqslant 1$, $u^* \in \partial g(u)$, $v^* \in \partial g(v)$ we must have $\langle u - v, u^* - v^* \rangle \geqslant 2\delta$. But, taking into account [10, Proposition 2.7], $u^* = 2u$, $v^* = 2v$, so that the above condition becomes $[\|w\|_3 \geqslant 1 \Rightarrow \|w\|_2 \geqslant \delta] \Leftrightarrow \delta \|w\|_3 \leqslant \|w\|_2$, which implies $L^2(0, 1) \subset L^3(0, 1)$, a contradiction.

5. APPENDIX

In the following proposition we collect some properties of convex functions over R.

PROPOSITION A.1. *Let* $f: R \to R \cup \{\infty\}$ *be a l.s.c. convex function. The following assertions hold*:

(i) *If* $t_0 \in \operatorname{dom} f$, *the map* $0 \neq t \in (\operatorname{dom} f - t_0) \to (f(t_0 + t) - f(t_0))/t$ *is nondecreasing. The above map is increasing for every* $t_0 \in \operatorname{dom} f$ *iff* f *is strictly convex.*

(ii) f'_+ *and* f'_- *are nondecreasing and finite on* $\operatorname{int}(\operatorname{dom} f)$; f *is strictly convex iff* $f'_+ (f'_-)$ *is increasing on* $\operatorname{int}(\operatorname{dom} f)$. *Moreover, for* $t_1 < t_2$,

$$f'_+(t_1) \leqslant f'_-(t_2) \leqslant f'_+(t_2)$$

and

$$\lim_{t \downarrow t_0} f'_+(t) = f'_+(t_0), \qquad \lim_{t \uparrow t_0} f'_+(t) = f'_-(t_0),$$

$$\lim_{t \downarrow t_0} f'_-(t) = f'_+(t_0), \qquad \lim_{t \uparrow t_0} f'_-(t) = f'_-(t_0).$$

(iii) *Let* $t_1, t_2 \in \operatorname{dom} f$, $t_1 < t_2$. *If for some* $\lambda_0 \in]0,1[$ *we have* $f(\lambda_0 t_1 + (1-\lambda_0) t_2) = \lambda_0 f(t_1) + (1-\lambda_0) f(t_2)$ *then* $f(\lambda t_1 + (1-\lambda) t_2) = \lambda f(t_1) + (1-\lambda) f(t_2)\ \forall \lambda \in [0,1]$, *so that*

$$f(t) = \frac{f(t_2) - f(t_1)}{t_2 - t_1}\, t + \frac{t_2 f(t_1) - t_1 f(t_2)}{t_2 - t_1}, \qquad \forall t \in [t_1, t_2].$$

(iv) $\partial f(t) = \{x \in R : f'_-(t) \leqslant x \leqslant f'_+(t)\} = [f'_-(t), f'_+(t)] \cap R.$

(v) *Let* $\varphi: R \to R \cup \{-\infty, +\infty\}$ *a nondecreasing function with* $\varphi(a) \in R$, $\varphi_+(t) = \lim_{\tau \downarrow t} \varphi(\tau)$, $\varphi_-(t) = \lim_{\tau \uparrow t} \varphi(\tau)$. *Then the function* $f: R \to R \cup \{\infty\}$, $f(x) = \int_a^x \varphi(t)\, dt$ *is a l.s.c. convex function with* $f'_- = \varphi_- \leqslant \varphi \leqslant \varphi_+ = f'_+$.

(vi) *Let* $t_0 \in \operatorname{dom} f$ *and* $\lambda \in]0,1[$ *and define* $\psi: (\operatorname{dom} f - t_0) \to R$, $\psi(t) = \lambda f(t_0 + t) - f(t_0 + \lambda t)$. *Then* ψ *is nonincreasing on* $I_1 =]-\infty, 0] \cap (\operatorname{dom} f - t_0)$ *and* ψ *is nondecreasing on* $I_2 = [0, \infty[\cap (\operatorname{dom} f - t_0)$. *If* f *is strictly convex then* ψ *is decreasing on* I_1 *and increasing on* I_2.

Proof. Properties (i)–(v) are known and can be found in [9]. Let us show (vi), which seems to be new. Let us prove that ψ is nondecreasing on I_2. So, let $0 < t_1 < t_2$, $t_2 \in \operatorname{dom} f - t_0$. Then we have

$$t_0 + \lambda t_1 < t_0 + t_1 < t_0 + t_2 \qquad \text{and} \qquad t_0 + \lambda t_1 < t_0 + \lambda t_2 < t_0 + t_2.$$

So, from the convexity of f it follows

$$f(t_0 + t_1) \leqslant \frac{t_2 - t_1}{t_2 - \lambda t_1} f(t_0 + \lambda t_1) + \frac{t_1 - \lambda t_1}{t_2 - \lambda t_1} f(t_0 + t_2),$$

$$f(t_0 + \lambda t_2) \leqslant \frac{t_2 - \lambda t_2}{t_2 - \lambda t_1} f(t_0 + \lambda t_1) + \frac{\lambda t_2 - \lambda t_1}{t_2 - \lambda t_1} f(t_0 + t_2),$$

so that

$$\lambda f(t_0 + t_1) + f(t_0 + \lambda t_2) \leqslant f(t_0 + \lambda t_1) + \lambda f(t_0 + t_2),$$

i.e., $\psi(t_1) \leqslant \psi(t_2)$, and the proof is complete. ∎

Let now $\varphi: R_+ \to \bar{R}_+$ be a nondecreasing function with the property that

$$\textit{there exists } x_0 > 0 \textit{ such that } 0 < \varphi_+(x_0) < \infty. \tag{A.1}$$

For such a function we put $\varphi_-(0) = 0$. To φ we associate the l.s.c. convex function

$$\psi: R_+ \to \bar{R}_+,\ \psi(x) = \int_0^x \varphi(t)\, dt. \tag{A.2}$$

From (A.1) we have $a = \sup(\operatorname{dom} \varphi) > 0$. ψ is a nondecreasing convex function, continuous on $[0, a[$, with the properties $\psi(0) = 0$, $[0, a[\subset \operatorname{dom} \psi \subset [0, a]$, $\lim_{t \uparrow a} \psi(t) = \psi(a)$ and there exist $x_0 > 0$ such that $0 < \psi(x_0) < \infty$ (therefore ψ is not an indicator function). Conversely if ψ has the above properties, then there exists a nondecreasing function φ (e.g., $\varphi = f'_+$ or f'_-) satisfying (A.1) such that ψ is given by (A.2). Note that if $a \in R$, $a \in \operatorname{dom} \psi$ iff $\varphi_-(a) < \infty$. For ψ as above consider $\psi^*: R_+ \to \bar{R}_+$, $\psi^*(x) = \sup\{tx - \psi(t): t \in R_+\}$. Taking the extension $\tilde{\psi}: R \to \bar{R}_+$, $\tilde{\psi}(x) = (1/p)\,|x|^p$ for $x < 0$ and $\tilde{\psi}(x) = \psi(x)$ for $x \geqslant 0$, where $p \in \,]1, \infty[$, $\tilde{\psi}$ is a l.s.c. convex function whose usual conjugate is $\tilde{\psi}^*: R \to \bar{R}_+$, $\tilde{\psi}^*(x) = (1/q)\,|x|^q$ for $x < 0$ and $\tilde{\psi}^*(x) = \psi^*(x)$ for $x \geqslant 0$, where $1/p + 1/q = 1$. Thus the general results in convex analysis apply also for that case (taking functions defined on R_+ with values in $\bar{R}_+$).

The following result seems to be new.

PROPOSITION A.2. *Let $\varphi: R_+ \to \bar{R}_+$ be a nondecreasing function with* $\operatorname{int}(\operatorname{dom} \varphi) \neq \varnothing$ *and* $\varphi \neq 0$, *and ψ defined by* (A.2).

(i) *Let*

$$\bar{\varphi}_+(\tau) = \max\{t \geqslant 0: \varphi_-(t) \leqslant \tau\} \text{ and } \bar{\varphi}_-(\tau) = \min\{t \geqslant 0: \varphi_+(t) \geqslant \tau\}.$$

Then $\psi_+^{*\prime} = \bar{\varphi}_+$ *and* $\psi_-^{*\prime} = \bar{\varphi}_-$.

(ii) *The following assertions are equivalent*: (a) $\varphi(t) = 0 \Rightarrow t = 0$; (b) $\psi(t) = 0 \Rightarrow t = 0$; (c) $\psi_+^{*\prime}(0) = 0$; (d) $\bar{\varphi}_+(0) = 0$.

Proof. (i) Since $\varphi_-(\varphi_+)$ is continuous at the left (right) (see Proposition A.1), in the definitions of $\bar{\varphi}_+(\bar{\varphi}_-)$ we have max(min), indeed. We have: $\tau \in [\psi'_-(t), \psi'_+(t)] \cap R = [\varphi_-(t), \varphi_+(t)] \cap R \Leftrightarrow \tau \in \partial\psi(t) \Leftrightarrow t \in \partial\psi^*(\tau) \Leftrightarrow t \in [\psi_-^{*\prime}(\tau), \psi_+^{*\prime}(\tau)] \cap R$. If $\bar{\varphi}_+(\tau) = \infty$ then $\varphi_-(t) \leqslant \tau\ \forall t \geqslant 0$ so that $\varphi_+(t) \leqslant \tau\ \forall t \geqslant 0$. Hence $R(\partial\psi) \subset [0, \tau]$ which implies $\operatorname{dom} \psi^* \subset \operatorname{cl} R(\partial\psi) \subset [0, \tau]$. Therefore $\psi_+^{*\prime}(\tau) = \infty$. Let now $t = \bar{\varphi}_+(\tau) < \infty$.

(a) If $\varphi_+(t) = \infty$ then $\partial\psi(t) = [\varphi_-(t), \infty[\subset R(\partial\psi) \subset \operatorname{dom} f^*$ so that $\operatorname{dom} f^* = R_+$. Therefore $\psi_+^{*\prime}(\tau) < \infty$.

(b) If $\varphi_+(t) < \infty$ then there exists $t' > t$ such that $\varphi_+(t') < \infty$, so that $\tau < \varphi_-(t') \leqslant \varphi_+(t')$. But $[\varphi_-(t'), \varphi_+(t')] = \partial\psi(t') \subset R(\partial\psi) \subset \operatorname{dom} f^*$, so that $\tau \in \operatorname{int}(\operatorname{dom} \psi^*)$; therefore $\psi_+^{*\prime}(\tau) < \infty$. Thus we may only take the case $t = \psi_+^{*\prime}(\tau) \in R (\Leftrightarrow \bar{\varphi}_+(\tau) \in R)$. Hence $\tau \in [\varphi_-(t), \varphi_+(t)]$, so that $\varphi_-(t) \leqslant \tau$, which implies $t = \psi_+^{*\prime}(\tau) \leqslant \bar{\varphi}_+(\tau)$. Let now $t = \bar{\varphi}_+(\tau) \in R$; then $\varphi_-(t) \leqslant \tau \leqslant \varphi_+(t)$, so that $t \in [\psi_-^{*\prime}(\tau), \psi_+^{*\prime}(\tau)]$, therefore $\bar{\varphi}_+(\tau) \leqslant \psi_+^{*\prime}(\tau)$. The proof for $\psi_-^{*\prime} = \bar{\varphi}_-$ is similar.

(ii) It is clear that (a) $\Leftrightarrow$ (b) and (c) $\Leftrightarrow$ (*d*). The proof for (*a*) $\Leftrightarrow$ (d) is immediate. ∎

Let now φ and ψ be as above, X a Banach space and define $f: X \to \bar{R}_+$ by

$$f(x) = \psi(\|x\|). \tag{A.3}$$

With the above notations and discussion we have $S(0, a) \subset \operatorname{dom} f \subset \bar{S}(0, a)$ and f is not an indicator function. To φ we associate the following *duality map*:

$$J_\varphi : X \to 2^{X^*}, \quad J_\varphi(x) = \{x^* \in X^*: \langle x, x^* \rangle = \|x\| \cdot \|x^*\|,$$
$$\varphi_-(\|x\|) \leqslant \|x^*\| \leqslant \varphi_+(\|x\|)\}. \tag{A.4}$$

When $\varphi: R_+ \to R_+$ is a *gauge map*, i.e., φ is increasing, continuous, $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$, J_φ is the usual duality map (see [8]). The following proposition collects some results concerning the relations between the properties of f, J_φ and those of X. These properties are surely known for φ a gauge map (see [8]).

PROPOSITION A.3. *Let φ, ψ, f, J_φ be as above and X a Banach space. Then*

(i) $$\partial f(x) = J_\varphi(x), \qquad \forall x \in X, \tag{A.5}$$

$$f^*(x^*) = \psi^*(\|x^*\|), \qquad \forall x^* \in X^*. \tag{A.6}$$

(ii) *The following conditions are equivalent*:

$$f \text{ is Gateaux differentiable on } \operatorname{int}(\operatorname{dom} f) = S(0, a), \tag{A.7}$$

$$J_\varphi \text{ is single-valued on } S(0, a), \tag{A.8}$$

$$\varphi \text{ is continuous on } [0, a[\text{ and } X \text{ is smooth} \tag{A.9}$$

and if X is reflexive,

$$\varphi \text{ is continuous on } [0, a[\text{ and } X^* \text{ is strictly convex.} \tag{A.10}$$

Moreover, if X is reflexive and one of the above conditions holds then

$$x_n, x \in S(0, a),\ x_n \to x \Rightarrow J_\varphi(x_n) \rightharpoonup J_\varphi(x). \tag{A.11}$$

(iii) *f is strictly convex iff ψ is strictly convex and X is strictly convex.*

(iv) *J_φ is onto iff $\lim_{t \to \infty} \varphi(t) = \infty$ and X is reflexive.*

Proof. (i) (A.5) is essentially [14, Theorem 1], and (A.6) is known and very easy to prove.

(ii) It is known (see [15]) that a convex function is Gateaux differentiable at a continuity point $\bar{x}$ iff $\partial f(\bar{x})$ is a singleton, so from (A.5) we have that (A.7) and (A.8) are equivalent.

(A.8) $\Rightarrow$ (A.9). For that, suppose J_φ is single-valued on $S(0, a)$. Let $t \in [0, a[$ and $x \in X$, $\|x\| = t$. If $t = 0$, taking into account (A.5), $J_\varphi(x) = \{x^* \in X^* : \|x^*\| \leqslant \varphi_+(0)\}$ so that $\varphi_+(0) = 0$; therefore φ is continuous at 0. Let now $t > 0$. From the Hahn–Banach theorem there is $x^* \in S^* = S_{X^*}$ such that $\langle x, x^* \rangle = \|x\|$. Hence $\varphi_-(t)x^*$, $\varphi_+(t)\, x^* \in J_\varphi(x)$, which, by (A.8), imply $\varphi_-(t) = \varphi_+(t)$. Therefore φ is continuous on $[0, a[$. Let now $x \in X \sim \{0\}$ and $x^* \in S^*$ such that $\langle x, x^* \rangle = \|x\|$; take $0 < a' < a$ such that $\varphi(a') > 0$. It is obvious that $\varphi(a')\, x^* \in J_\varphi((a'/\|x\|)\, x)$. Since $J_\varphi((a'/\|x\|)\, x)$ is a singleton it follows that X is smooth. The converse implication is obvious. If X is reflexive then (see [11]) X is smooth iff X^* is strictly convex, so that in this case (A.9) $\Leftrightarrow$ (A.10). Suppose X is reflexive and show that (A.9) $\Rightarrow$ (A.11). Let x_n, $x \in S(0, a)$, $x_n \to x$; then $\|J_\varphi(x_n)\| = \varphi(\|x_n\|) \to \varphi(\|x\|) = \|J_\varphi(x)\|$. Therefore $\{J_\varphi(x_n)\}$ is bounded so that, by the reflexivity of X, there exists a subsequence $\{J_\varphi(x_{n_k})\}$ such that $J_\varphi(x_{n_k}) \rightharpoonup x^*$. But $\langle x_{n_k}, J_\varphi(x_{n_k}) \rangle = \|x_{n_k}\|\, \varphi(\|x_{n_k}\|)$; taking the limit we obtain $\langle x, x^* \rangle = \|x\|\, \varphi(\|x\|)$, which implies $\varphi(\|x\|) \leqslant \|x^*\|$. On the other hand, from $J_\varphi(x_{n_k}) \rightharpoonup x^*$ it follows $\|x^*\| \leqslant \liminf_{k \to \infty} \|J_\varphi(x_{n_k})\| = \varphi(\|x\|)$. Hence $\|x^*\| = \varphi(\|x\|)$ which, together with $\langle x, x^* \rangle = \|x\|\, \varphi(\|x\|)$, gives $x^* = J_\varphi(x)$. Thus $J_\varphi(x_n) \rightharpoonup J_\varphi(x)$.

(iii) Suppose f is strictly convex. Therefore

$$\begin{gathered} \forall x, y \in \operatorname{dom} f, \quad x \neq y, \quad \forall \lambda \in]0, 1[: \\ f(\lambda x + (1 - \lambda) y) < \lambda f(x) + (1 - \lambda) f(y). \end{gathered} \tag{A.12}$$

Taking $x = tu$, $y = su$, $t, s \in \operatorname{dom} \psi$, $t \neq s$ and $u \in S$ in (A.12), we obtain ψ is strictly convex. Taking now $0 < t = \|x\| = \|y\| \in \operatorname{dom} \psi$, $x \neq y$, from (A.12) we obtain $\psi(\|\lambda x + (1 - \lambda) y\|) < \psi(t)$ so that $\|\lambda x + (1 - \lambda) y\| < \|x\| = \|y\|$ $\forall \lambda \in]0, 1[$, which shows that X is strictly convex.

Suppose now that ψ and X are strictly convex and let $x, y \in \operatorname{dom} f$, $x \neq y$, $\lambda \in]0, 1[$. Therefore, $\|x\|$, $\|y\| \in \operatorname{dom} \psi$. If $\|x\| \neq \|y\|$ then

$$\begin{aligned} f(\lambda x + (1 - \lambda) y) &= \psi(\|\lambda x + (1 - \lambda) y\|) \leqslant \psi(\lambda\, \|x\| + (1 - \lambda)\, \|y\|) \\ &< \lambda \psi(\|x\|) + (1 - \lambda)\, \psi(\|y\|) = \lambda f(x) + (1 - \lambda) f(y). \end{aligned}$$

If $\|x\| = \|y\|$, then from the strict convexity of X we have $\|\lambda x + (1 - \lambda) y\| < \lambda\, \|x\| + (1 - \lambda)\, \|y\|$, so that

$$\begin{aligned} f(\lambda x + (1 - \lambda) y) &= \psi(\|\lambda x + (1 - \lambda) y\|) < \psi(\lambda\, \|x\| + (1 - \lambda)\, \|y\|) \\ &< \lambda \psi(\|x\|) + (1 - \lambda)\, \psi(\|y\|) = \lambda f(x) + (1 - \lambda)\, f(y). \end{aligned}$$

Therefore f is strictly convex.

(iv) Suppose J_φ is onto and let $L = \lim_{t\to\infty} \varphi(t)$. If $x^* \in R(J_\varphi)$ then $\|x^*\| \leqslant L$, so that we must have $L = \infty$. By a theorem of James [11] X is reflexive iff $\forall x^* \in X^* \sim \{0\}$ $\exists x \in S$ such that $\langle x, x^* \rangle = \|x^*\|$. Let $x^* \in X^* \sim \{0\}$; there exist $t > 0$ and $\lambda > 0$ such that

$$\varphi_-(t) \leqslant \lambda \|x^*\| \leqslant \varphi_+(t), \varphi_+(0) < \lambda \|x^*\|. \tag{A.13}$$

Since J_φ is onto, there exists $x \in X$ such that $\lambda x^* \in J_\varphi(x)$; hence $\langle x, x^* \rangle = \|x\| \|x^*\|$, $\varphi_-(\|x\|) \leqslant \lambda \|x^*\| \leqslant \varphi_+(\|x\|)$. By (A.13), $x \neq 0$. The converse implication is immediate. ▮

PROPOSITION A.4. *Let X be a Banach space and $f: X \to R \cup \{\infty\}$ a proper l.s.c. convex function. Then*

(i) $\liminf_{\|x\|\to\infty}(f(x)/\|x\|^p) > 0$ *iff* $\limsup_{\|x^*\|\to\infty}(f^*(x^*)/\|x^*\|^q) < \infty$, *where* $p, q > 1$, $1/p + 1/q = 1$.

(ii) *If X is reflexive then* $\lim_{\|x\|\to\infty}(f(x)/\|x\|) = \infty$ *iff ∂f^* is bounded on bounded sets.*

(iii) $\liminf_{\|x\|\to\infty}((f(x) - \langle x, x^* \rangle)/\|x\|) > 0$ *iff* $x^* \in \operatorname{int}(\operatorname{dom} f^*)$.

Proof. Since f is a proper l.s.c. convex function, there exist $x_0^* \in X^*$, $\alpha \in R$ such that $f(x) \geqslant \langle x, x_0^* \rangle + \alpha$ $\forall x \in X$.

(i) Suppose $\liminf_{\|x\|\to\infty}(f(x)/\|x\|^p) > 0$. Then for some m, $M > 0$ we have

$$\|x\| \geqslant M \Rightarrow f(x) \geqslant m \cdot \|x\|^p.$$

Take $x^* \in X^*$; then

$$\begin{aligned} f^*(x^*) &= \sup\{\langle x, x^* \rangle - f(x): x \in X\} \\ &= \max\{\sup\{\langle x, x^* \rangle - f(x): \|x\| \leqslant M\}, \\ &\qquad \sup\{\langle x, x^* \rangle - f(x): \|x\| \geqslant M\}\} \\ &\leqslant \max\{\sup\{\langle x, x^* \rangle - \langle x, x_0^* \rangle - \alpha; \|x\| \leqslant M\}, \\ &\qquad \sup\{\langle x, x^* \rangle - m\|x\|^p: x \in X\}\} \\ &= \max\left\{M\|x^* - x_0{}^*\| - \alpha, \frac{(mp)^{1-q}}{q}\|x^*\|^q\right\}. \end{aligned}$$

Since $q > 1$, it is obvious from the above inequality that $\limsup_{\|x^*\|\to\infty} (f^*(x^*)/\|x^*\|^q) < \infty$. In a similar way the converse implication can be proved.

(ii) This is nothing else but [10, Proposition 2.5].

(iii) Consider the case $x^* = 0$. Suppose $\liminf_{\|x\|\to\infty}(f(x)/\|x\|) > 0$.

Take $m, M > 0$ such that $\|x\| \geqslant M \Rightarrow f(x) \geqslant m\|x\|$. As in (i) we find

$$f^*(x^*) \leqslant \max\{M\|x^* - x_0^*\| - \alpha, I_{\bar{S}^*(0,m)}(x^*)\},$$

so that $S^*(0, m) \subset \operatorname{dom} f^*$.

Conversely, suppose $0 \in \operatorname{int}(\operatorname{dom} f^*)$. Then f^* is Lipschitz at 0 (see, e.g., [16, Proposition 2]). Therefore there exist $m, M > 0$ such that

$$f^*(x^*) - f^*(0) \leqslant M\|x^*\|, \qquad \forall x^* \in \bar{S}(0, m).$$

For $\|x\| \geqslant M$,

$$\begin{aligned} f(x) &\geqslant \sup\{\langle x, x^*\rangle - f^*(x^*): \|x^*\| \leqslant m\} \\ &\geqslant \sup\{\langle x, x^*\rangle - M\|x^*\| - f^*(0): \|x^*\| \leqslant m\} \\ &= m(\|x\| - M) - f^*(0). \end{aligned}$$

Hence $\liminf_{\|x\|\to\infty}(f(x)/\|x\|) > 0$. ∎

The following result seems to be new.

PROPOSITION A.5. *Let $f: R_+ \to \bar{R}_+$ be a nondecreasing function with $f(x) = 0 \Leftrightarrow x = 0$, and $g = \overline{\text{co}}\, f$. The following assertions are equivalent*:

(i) $\exists x > 0$ *such that* $g(x) > 0$,

(ii) $g(x) > 0\ \forall x > 0$,

(iii) $\liminf_{x\to\infty}(f(x)/x) > 0$.

Moreover, if one of the above conditions holds, then for $p \geqslant 1$ we have

(iv) $\liminf_{x\to\infty}(f(x)/x^p) \geqslant \liminf_{x\to\infty}(g(x)/x^p) \geqslant (1/3^{p-1}) \cdot \liminf_{x\to\infty}(f(x)/x^p)$ *and* $\lim_{x\to\infty}(g(x)/x) = \liminf_{x\to\infty}(f(x)/x)$;

(v) *if* $\liminf_{x\downarrow 0}(f(x)/x^p) > 0$ *then* $\liminf_{x\downarrow 0}(g(x)/x^p) > 0$.

Proof. It is obvious that $0 \leqslant g \leqslant f$, so that $g(0) = 0$. The implication (ii) $\Rightarrow$ (*i*) is clear.

(iii) $\Rightarrow$ (ii). Put $0 < a = \liminf_{x\to\infty}(f(x)/x)$ and let $x \geqslant 0$ with $g(x) = 0$; hence $(x, 0) \in \operatorname{epi} g = \overline{\text{co}}(\operatorname{epi} f) \subset R^2$. Using Helly's theorem [9], for every $i = 1, 2, 3$ there exist the sequences (λ_n^i), (x_n^i), $(t_n^i) \subset R_+$ with $\lambda_n^1 + \lambda_n^2 + \lambda_n^3 = 1$, $t_n^i \geqslant f(x_n^i)$ such that $\sum_{i=1}^3 \lambda_n^i x_n^i \to x$, and $\sum_{i=1}^3 \lambda_n^i t_n^i \to g(x)$. W.l.o.g. we can suppose that

$$\begin{aligned} &x_n^i \to x_i \in \bar{R}_+,\ t_n^i \to t^i \in \bar{R}_+,\ t_n^i \geqslant f(x_n^i),\ \lambda_n^i \to \lambda^i,\ \lambda^1 + \lambda^2 + \lambda^3 = 1, \\ &\lambda_n^i x_n^i \to y^i \in R_+,\ y^1 + y^2 + y^3 = x \text{ and } \lambda_n^i t_n^i \to \tau^i \in R_+, \qquad \text{(A.14)} \\ &\tau^1 + \tau^2 + \tau^3 = g(x). \end{aligned}$$

Since $g(x)=0$, it follows that $\tau^1=\tau^2=\tau^3=0$. If $\lambda^i>0$, since $\lambda_n^i t_n^i \to 0$, it follows that $t^i=0$ and so $f(x_n^i)\to 0$, so that, by the hypothesis, $x_n^i \to x^i=0$, which in its turn implies $y^i=0$. If $\lambda^i=0$ and $y^i>0$ then $x_n^i\to x^i=\infty$; let $0<a'<a$. Thus there exists $n'\in N$ such that for $n\geqslant n'$, $f(x_n^i)\geqslant a'x_n^i$, so that $t_n^i\geqslant a'x_n^i$. Therefore, for $n\geqslant n'$, $\lambda_n^i t_n^i\geqslant a'\lambda_n^i x_n^i$; taking the limit we obtain $0=\tau^i\geqslant a'y^i$ which contradicts $y^i>0$. Hence $y^1=y^2=y^3=0$, so that $x=0$.

(i) $\Rightarrow$ (iii). Suppose $\liminf_{x\to\infty} f(x)/x=0$: therefore, for a sequence $(x_n)\subset R_+$ with $x_n\to\infty$, we have $f(x_n)=\alpha_n x_n$, $\alpha_n\to 0$. Let $x\in R_+^*$; there exists $n_x\in N$ such that for $n\geqslant n_x$ we have $x_n\geqslant x$. Let $\lambda_n=x/x_n\in\,]0,1[$ for $n\geqslant n_x$; then $\lambda_n(x_n,f(x_n))=(\lambda_n x_n,\lambda_n f(x_n))=(x,\alpha_n x)\in \operatorname{co}(\operatorname{epi} f)\subset \operatorname{epi} g$, which implies $(x,0)\in\operatorname{epi} g$. Therefore $g(x)=0$ for every $x\in R_+^*$, a contradiction.

Suppose now (iii) is verified. Since $g(x)=0\Leftrightarrow x=0$ and g is convex, it follows g is increasing on dom g. Because $0\leqslant g\leqslant f$, we have, for $p\geqslant 1$,

$$\liminf_{x\to\infty}\frac{g(x)}{x^p}\leqslant\liminf_{x\to\infty}\frac{f(x)}{x^p}. \tag{A.15}$$

If $\operatorname{dom} g\neq R_+$, (iv) is obvious, so that suppose $\operatorname{dom} g=R_+$. Let $0<a'<a=\liminf_{x\to\infty}(f(x)/x^p)$ [if $a=0$, from (A.15), (iv) follows]; hence for some $M>0$ we have $f(x)\geqslant a'x^p$ $\forall x\geqslant M$. Let $x>M$ and the sequences (λ_n^i), (x_n^i), $(t_n^i)\subset R_+$ satisfying (A.14). It follows that $x^i>M$ for at least one i. If $x^i>M$ then $t_n^i\geqslant a'(x_n^i)^p$ for n sufficiently large, so that $\lambda_n^i t_n^i\geqslant a'\lambda_n^i(x_n^i)^p\geqslant a'(\lambda_n^i x_n^i)^p$. Therefore $\tau^i\geqslant a'(y^i)^p$. So, in the case $x^1>M$, $x^2,x^3\leqslant M$ we have $y_2+y_3\leqslant M$, so that $g(x)=\tau^1+\tau^2+\tau^3\geqslant\tau^1\geqslant a'(y^1)^p=a'(x-(y^2+y^3))^p\geqslant a'(x-M)^p$. Hence $g(x)/x^p\geqslant a'(1-M/x)^p$. Similarly, in the case x^1, $x^2>M$, $x^3\leqslant M$ we obtain $g(x)/x^p\geqslant(a'/2^{p-1})(1-M/x)^p$ and for $x^1,x^2,x^3>M$ we have $g(x)/x^p\geqslant a'/3^{p-1}$. Therefore $\liminf_{x\to\infty}(g(x)/x^p)\geqslant a'/3^{p-1}$ for every $a'<a$. Thus the first part of (iv) is true. For $p=1$ we obtain

$$\liminf_{x\to\infty}\frac{f(x)}{x}=\liminf_{x\to\infty}\frac{g(x)}{x}=\lim_{x\to\infty}\frac{g(x)}{x},$$

the last limit exists because g is convex [see also Proposition A.1(i)]. The proof of (v) is similar. ∎

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